Chapter 6 Overview

One of the early accomplishments of calculus was predicting the future position of a planet from its present position and velocity. Today this is just one of a number of occasions on which we deduce everything we need to know about a function from one of its known values and its rate of change. From this kind of information, we can tell how long a sample of radioactive polonium will last; whether, given current trends, a population will grow or become extinct; and how large major league baseball salaries are likely to be in the year 2010. In this chapter, we examine the analytic, graphical, and numerical techniques on which such predictions are based.

6.1 Slope Fields and Euler's Method

What you'll learn about

- Differential Equations
- Slope Fields
- Euler's Method

. . . and why

Differential equations have always been a prime motivation for the study of calculus and remain so to this day.

Differential Equations

We have already seen how the discovery of calculus enabled mathematicians to solve problems that had befuddled them for centuries because the problems involved moving objects. Leibniz and Newton were able to model these problems of motion by using equations involving derivatives—what we call *differential equations* today, after the notation of Leibniz. Much energy and creativity has been spent over the years on techniques for solving such equations, which continue to arise in all areas of applied mathematics.

DEFINITION Differential Equation

An equation involving a derivative is called a **differential equation.** The **order of a differential equation** is the order of the highest derivative involved in the equation.

EXAMPLE 1 Solving a Differential Equation

Find all functions y that satisfy $dy/dx = \sec^2 x + 2x + 5$.

SOLUTION

We first encountered this sort of differential equation (called *exact* because it gives the derivative exactly) in Chapter 4. The solution can be any antiderivative of $\sec^2 x + 2x + 5$, which can be any function of the form $y = \tan x + x^2 + 5x + C$. That family of functions is the *general* solution to the differential equation. ***Now try Exercise 1.***

Notice that we cannot find a unique solution to a differential equation unless we are given further information. If the general solution to a first-order differential equation is continuous, the only additional information needed is the value of the function at a single point, called an *initial condition*. A differential equation with an initial condition is called an *initial value problem*. It has a unique solution, called the *particular solution* to the differential equation.

EXAMPLE 2 Solving an Initial Value Problem

Find the particular solution to the equation $dy/dx = e^x - 6x^2$ whose graph passes through the point $(1, 0)$.

SOLUTION

The general solution is $y = e^x - 2x^3 + C$. Applying the initial condition, we have $0 = e - 2 + C$, from which we conclude that $C = 2 - e$. Therefore, the particular solution is $y = e^x - 2x^3 + 2 - e$. ***Now try Exercise 13.***

An initial condition determines a particular solution by requiring that a solution curve pass through a given point. If the curve is continuous, this pins down the solution on the entire domain. If the curve is discontinuous, the initial condition only pins down the continuous *piece of the curve* that passes through the given point. In this case, the domain of the solution must be specified.

EXAMPLE 3 Handling Discontinuity in an Initial Value Problem

Find the particular solution to the equation $dy/dx = 2x - \sec^2 x$ whose graph passes through the point (0, 3).

SOLUTION

The general solution is $y = x^2 - \tan x + C$. Applying the initial condition, we have $3 = 0 - 0 + C$, from which we conclude that $C = 3$. Therefore, the particular solution is $y = x^2 - \tan x + 3$. Since the point (0, 3) only pins down the continuous piece of the general solution over the interval $(-\pi/2, \pi/2)$, we add the domain stipulation $-\pi/2 < x < \pi/2$.

Now try Exercise 15.

Sometimes we are unable to find an antiderivative to solve an initial value problem, but we can still find a solution using the Fundamental Theorem of Calculus.

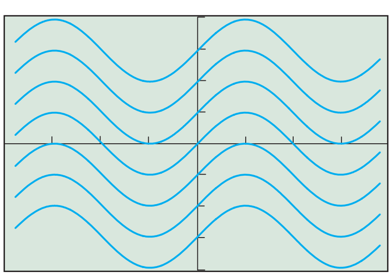
EXAMPLE 4 Using the Fundamental Theorem to Solve an Initial Value Problem

Find the solution to the differential equation $f'(x) = e^{-x^2}$ for which $f(7) = 3$.

SOLUTION

This almost seems too simple, but $f(x) = \int_7^x e^{-t^2}\,dt + 3$ has both of the necessary properties! Clearly, $f(7) = \int_7^7 e^{-t^2}\,dt + 3 = 0 + 3 = 3$, and $f'(x) = e^{x^2}$ by the Fundamental Theorem. The integral form of the solution in Example 4 might seem less desirable than the explicit form of the solutions in Examples 2 and 3, but (thanks to modern technology) it does enable us to find $f(x)$ for any x. For example, $f(-2) = \int_7^{-2} e^{-t^2}\,dt + 3 =$ fnInt $(e\text{^}(-t^2), t, 7, -2) + 3 \approx 1.2317$.

Now try Exercise 21.



[–2π, 2π] by [–4, 4]

Figure 6.1 A graph of the family of functions $Y_1 = \sin(x) + L_1$, where $L_1 = \{-3, -2, -1, 0, 1, 2, 3\}$. This graph shows some of the functions that satisfy the differential equation $dy/dx = \cos x$. (Example 5)

EXAMPLE 5 Graphing a General Solution

Graph the family of functions that solve the differential equation $dy/dx = \cos x$.

SOLUTION

Any function of the form $y = \sin x + C$ solves the differential equation. We cannot graph them all, but we can graph enough of them to see what a family of solutions would look like. The command $\{-3, -2, -1, 0, 1, 2, 3,\} \rightarrow L_1$ stores seven values of C in the list L_1. Figure 6.1 shows the result of graphing the function $Y_1 = \sin(x) + L_1$.

Now try Exercises 25–28.

Notice that the graph in Figure 6.1 consists of a family of parallel curves. This should come as no surprise, since functions of the form $\sin(x) + C$ are all vertical translations of the basic sine curve. It might be less obvious that we could have predicted the appearance of this family of curves from *the differential equation itself*. Exploration 1 gives you a new way to look at the solution graph.

EXPLORATION 1 Seeing the Slopes

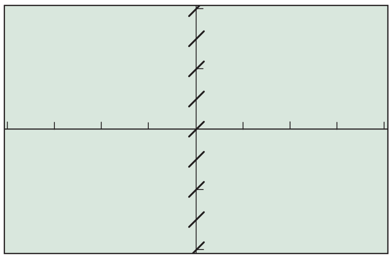
Figure 6.1 shows the general solution to the exact differential equation $dy/dx = \cos x$.

1. Since $\cos x = 0$ at odd multiples of $\pi/2$, we should "see" that $dy/dx = 0$ at the odd multiples of $\pi/2$ in Figure 6.1. Is that true? How can you tell?
2. Algebraically, the y-coordinate does not affect the value of $dy/dx = \cos x$. Why not?
3. Does the graph show that the y-coordinate does not affect the value of dy/dx? How can you tell?
4. According to the differential equation $dy/dx = \cos x$, what should be the slope of the solution curves when $x = 0$? Can you see this in the graph?
5. According to the differential equation $dy/dx = \cos x$, what should be the slope of the solution curves when $x = \pi$? Can you see this in the graph?
6. Since $\cos x$ is an even function, the slope at any point should be the same as the slope at its reflection across the y-axis. Is this true? How can you tell?

Exploration 1 suggests the interesting possibility that we could have produced the family of curves in Figure 6.1 without even solving the differential equation, simply by looking carefully at slopes. That is exactly the idea behind *slope fields*.

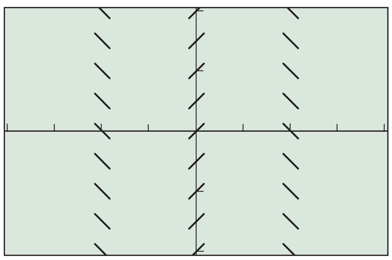
Slope Fields

Suppose we want to produce Figure 6.1 without actually solving the differential equation $dy/dx = \cos x$. Since the differential equation gives the *slope* at any point (x, y), we can use that information to draw a small piece of the linearization at that point, which (thanks to local linearity) approximates the solution curve that passes through that point. Repeating that process at many points yields an approximation of Figure 6.1 called a slope field. Example 6 shows how this is done.



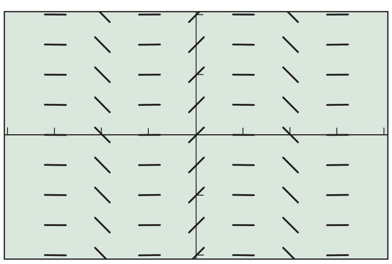
$[-2\pi, 2\pi]$ by $[-4, 4]$

(a)



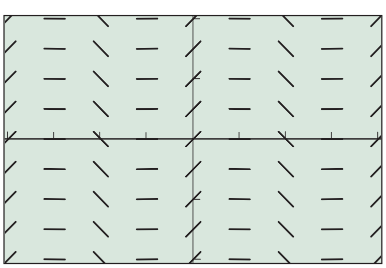
$[-2\pi, 2\pi]$ by $[-4, 4]$

(b)



$[-2\pi, 2\pi]$ by $[-4, 4]$

(c)



$[-2\pi, 2\pi]$ by $[-4, 4]$

(d)

Figure 6.2 The steps in constructing a slope field for the differential equation $dy/dx = \cos x$. (Example 6)

EXAMPLE 6 Constructing a Slope Field

Construct a slope field for the differential equation $dy/dx = \cos x$.

SOLUTION

We know that the slope at any point $(0, y)$ will be $\cos 0 = 1$, so we can start by drawing tiny segments with slope 1 at several points along the y-axis (Figure 6.2a). Then, since the slope at any point (π, y) or $(-\pi, y)$ will be -1, we can draw tiny segments with slope -1 at several points along the vertical lines $x = \pi$ and $x = -\pi$ (Figure 6.2b). The slope at all odd multiples of $\pi/2$ will be zero, so we draw tiny horizontal segments along the lines $x = \pm\pi/2$ and $x = \pm 3\pi/2$ (Figure 6.2c). Finally, we add tiny segments of slope 1 along the lines $x = \pm 2\pi$ (Figure 6.2d).

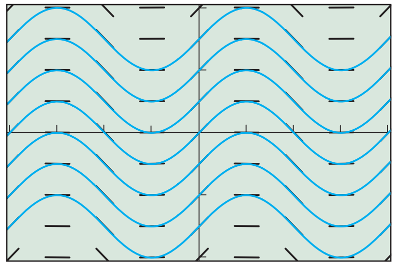
Now try Exercise 29.

To illustrate how a family of solution curves conforms to a slope field, we superimpose the solutions in Figure 6.1 on the slope field in Figure 6.2d. The result is shown in Figure 6.3 on the next page.

We could get a smoother-looking slope field by drawing shorter line segments at more points, but that can get tedious. Happily, the algorithm is simple enough to be programmed into a graphing calculator. One such program, using a lattice of 150 sample points, produced in a matter of seconds the graph in Figure 6.4 on the next page.

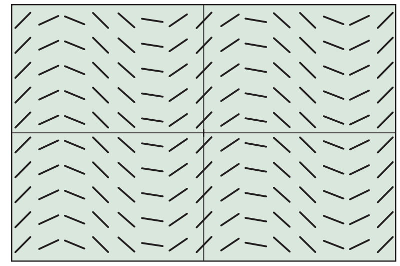
Differential Equation Mode

If your calculator has a *differential equation mode* for graphing, it is intended for graphing slope fields. The usual "Y=" turns into a "$dy/dx =$" screen, and you can enter a function of x and/or y. The grapher draws a slope field for the differential equation when you press the GRAPH button.



$[-2\pi, 2\pi]$ by $[-4, 4]$

Figure 6.3 The graph of the general solution in Figure 6.1 conforms nicely to the slope field of the differential equation. (Example 6)



$[-2\pi, 2\pi]$ by $[-4, 4]$

Figure 6.4 A slope field produced by a graphing calculator program.

It is also possible to produce slope fields for differential equations that are not of the form $dy/dx = f(x)$. We will study analytic techniques for solving certain types of these nonexact differential equations later in this chapter, but you should keep in mind that you can graph the general solution with a slope field even if you cannot find it analytically.

Can We Solve the Differential Equation in Example 7?

Although it looks harmless enough, the differential equation $dy/dx = x + y$ is not easy to solve until you have seen how it is done. It is an example of a *first-order linear differential equation,* and its general solution is

$$y = Ce^x - x - 1$$

(which you can easily check by verifying that $dy/dx = x + y$). We will defer the analytic solution of such equations to a later course.

EXAMPLE 7 Constructing a Slope Field for a Nonexact Differential Equation

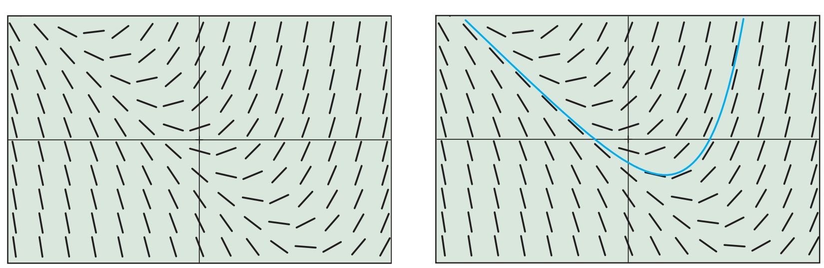
Use a calculator to construct a slope field for the differential equation $dy/dx = x + y$ and sketch a graph of the particular solution that passes through the point (2, 0).

SOLUTION

The calculator produces a graph like the one in Figure 6.5a. Notice the following properties of the graph, all of them easily predictable from the differential equation:

1. The slopes are zero along the line $x + y = 0$.
2. The slopes are -1 along the line $x + y = -1$.
3. The slopes get steeper as x increases.
4. The slopes get steeper as y increases.

The particular solution can be found by drawing a smooth curve through the point (2, 0) that follows the slopes in the slope field, as shown in Figure 6.5b.



$[-4.7, 4.7]$ by $[-3.1, 3.1]$
(a)

$[-4.7, 4.7]$ by $[-3.1, 3.1]$
(b)

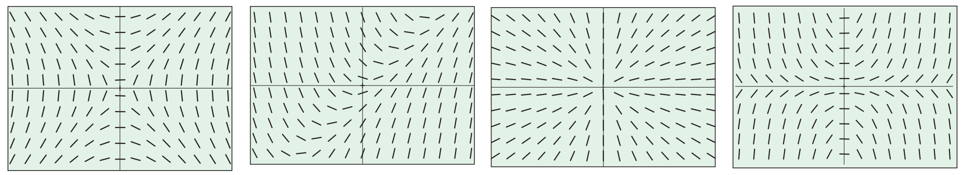
Figure 6.5 (a) A slope field for the differential equation $dy/dx = x + y$, and (b) the same slope field with the graph of the particular solution through (2, 0) superimposed. (Example 7)

Now try Exercise 35.

EXAMPLE 8 Matching Slope Fields with Differential Equations

Use slope analysis to match each of the following differential equations with one of the slope fields (a) through (d). (Do not use your graphing calculator.)

1. $\dfrac{dy}{dx} = x - y$ **2.** $\dfrac{dy}{dx} = xy$ **3.** $\dfrac{dy}{dx} = \dfrac{x}{y}$ **4.** $\dfrac{dy}{dx} = \dfrac{y}{x}$



(a) (b) (c) (d)

SOLUTION

To match Equation 1, we look for a graph that has zero slope along the line $x - y = 0$. That is graph (b).

To match Equation 2, we look for a graph that has zero slope along both axes. That is graph (d).

To match Equation 3, we look for a graph that has horizontal segments when $x = 0$ and vertical segments when $y = 0$. That is graph (a).

To match Equation 4, we look for a graph that has vertical segments when $x = 0$ and horizontal segments when $y = 0$. That is graph (c). ***Now try Exercise 39.***

Euler's Method

In Example 7 we graphed the particular solution to an initial value problem by first producing a slope field and then finding a smooth curve through the slope field that passed through the given point. In fact, we could have graphed the particular solution directly, by starting at the given point and piecing together little line segments to build a continuous approximation of the curve. This clever application of local linearity to graph a solution without knowing its equation is called **Euler's Method.**

Euler's Method For Graphing a Solution to an Initial Value Problem

1. Begin at the point (x, y) specified by the initial condition. This point will be on the graph, as required.
2. Use the differential equation to find the slope dy/dx at the point.
3. Increase x by a small amount Δx. Increase y by a small amount Δy, where $\Delta y = (dy/dx)\Delta x$. This defines a new point $(x + \Delta x, y + \Delta y)$ that lies along the linearization (Figure 6.6).
4. Using this new point, return to step 2. Repeating the process constructs the graph to the right of the initial point.
5. To construct the graph moving to the left from the initial point, repeat the process using negative values for Δx.

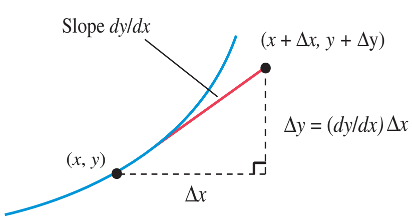


Figure 6.6 How Euler's Method moves along the linearization at the point (x, y) to define a new point $(x + \Delta x, y + \Delta y)$. The process is then repeated, starting with the new point.

We illustrate the method in Example 9.

EXAMPLE 9 Applying Euler's Method

Let f be the function that satisfies the initial value problem in Example 6 (that is, $dy/dx = x + y$ and $f(2) = 0$). Use Euler's method and increments of $\Delta x = 0.2$ to approximate $f(3)$.

SOLUTION

We use Euler's Method to construct an approximation of the curve from $x = 2$ to $x = 3$, pasting together five small linearization segments (Figure 6.7). Each segment will extend from a point (x, y) to a point $(x + \Delta x, y + \Delta y)$, where $\Delta x = 0.2$ and $\Delta y = (dy/dx)\Delta x$. The following table shows how we construct each new point from the previous one.

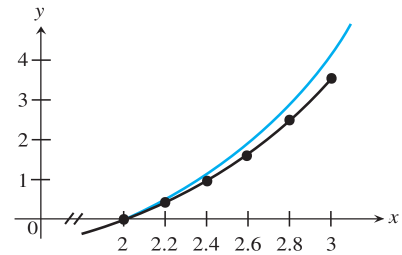


Figure 6.7 Euler's Method is used to construct an approximate solution to an initial value problem between $x = 2$ and $x = 3$. (Example 9)

(x, y)	$dy/dx = x + y$	Δx	$\Delta y = (dy/dx)\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 0)	2	0.2	0.4	(2.2, 0.4)
(2.2, 0.4)	2.6	0.2	0.52	(2.4, 0.92)
(2.4, 0.92)	3.32	0.2	0.664	(2.6, 1.584)
(2.6, 1.584)	4.184	0.2	0.8368	(2.8, 2.4208)
(2.8, 2.4208)	5.2208	0.2	1.04416	(3, 3.46496)

Euler's Method leads us to an approximation $f(3) \approx 3.46496$, which we would more reasonably report as $f(3) \approx 3.465$. ***Now try Exercise 41.***

You can see from Figure 6.7 that Euler's Method leads to an underestimate when the curve is concave up, just as it will lead to an overestimate when the curve is concave down. You can also see that the error increases as the distance from the original point increases. In fact, the true value of $f(3)$ is about 4.155, so the approximation error is about 16.6%. We could increase the accuracy by taking smaller increments; a reasonable option if we have a calculator program to do the work. For example, 100 increments of 0.01 give an estimate of 4.1144, cutting the error to about 1%.

EXAMPLE 10 Moving Backward with Euler's Method

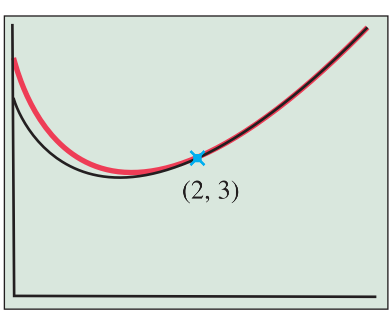
If $dy/dx = 2x - y$ and if $y = 3$ when $x = 2$, use Euler's Method with five equal steps to approximate y when $x = 1.5$.

SOLUTION

Starting at $x = 2$, we need five equal steps of $\Delta x = -0.1$.

(x, y)	$dy/dx = 2x - y$	Δx	$\Delta y = (dy/dx)\Delta x$	$(x + \Delta x, y + \Delta y)$
(2, 3)	1	−0.1	−0.1	(1.9, 2.9)
(1.9, 2.9)	0.9	−0.1	−0.09	(1.8, 2.81)
(1.8, 2.81)	0.79	−0.1	−0.079	(1.7, 2.731)
(1.7, 2.731)	0.669	−0.1	−0.0669	(1.6, 2.6641)
(1.6, 2.6641)	0.5359	−0.1	−0.05359	(1.5, 2.61051)

The value at $x = 1.5$ is approximately 2.61. (The actual value is about 2.649, so the percentage error in this case is about 1.4%.) ***Now try Exercise 45.***



[0, 4] by [0, 6]

Figure 6.8 A grapher program using Euler's Method and increments of 0.1 produced this approximation to the solution curve for the initial value problem in Example 10. The actual solution curve is shown in red.

If we program a grapher to do the work of finding the points, Euler's Method can be used to graph (approximately) the solution to an initial value problem without actually solving it. For example, a graphing calculator program starting with the initial value problem in Example 9 produced the graph in Figure 6.8 using increments of 0.1. The graph of the actual solution is shown in red. Notice that Euler's Method does a better job of approximating the curve when the curve is nearly straight, as should be expected.

Euler's Method is one example of a *numerical method* for solving differential equations. The table of values is the *numerical solution*. The analysis of error in a numerical solution and the investigation of methods to reduce it are important, but appropriate for a more advanced course (which would also describe more accurate numerical methods than the one shown here).

Quick Review 6.1

In Exercises 1–8, determine whether or not the function y satisfies the differential equation.

1. $\frac{dy}{dx} = y \qquad y = e^x$ Yes
2. $\frac{dy}{dx} = 4y \qquad y = e^{4x}$ Yes
3. $\frac{dy}{dx} = 2xy \qquad y = x^2e^x$ No
4. $\frac{dy}{dx} = 2xy \qquad y = e^{x^2}$ Yes
5. $\frac{dy}{dx} = 2xy \qquad y = e^{x^2} + 5$ No
6. $\frac{dy}{dx} = \frac{1}{y} \qquad y = \sqrt{2x}$ Yes
7. $\frac{dy}{dx} = y \tan x \qquad y = \sec x$ Yes
8. $\frac{dy}{dx} = y^2 \qquad y = x^{-1}$ No

In Exercises 9–12, find the constant C.

9. $y = 3x^2 + 4x + C$ and $y = 2$ when $x = 1$ $\quad -5$
10. $y = 2\sin x - 3\cos x + C$ and $y = 4$ when $x = 0$ $\quad 7$
11. $y = e^{2x} + \sec x + C$ and $y = 5$ when $x = 0$ $\quad 3$
12. $y = \tan^{-1} x + \ln(2x - 1) + C$ and $y = \pi$ when $x = 1$ $\quad 3\pi/4$

Section 6.1 Exercises

In Exercises 1–10, find the general solution to the exact differential equation.

1. $\frac{dy}{dx} = 5x^4 - \sec^2 x \quad y = x^5 - \tan x + C$
2. $\frac{dy}{dx} = \sec x \tan x - e^x \quad y = \sec x - e^x + C$
3. $\frac{dy}{dx} = \sin x - e^{-x} + 8x^3 \quad y = -\cos x + e^{-x} + 2x^4 + C$
4. $\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x^2} \; (x > 0) \quad y = \ln x + x^{-1} + C$
5. $\frac{dy}{dx} = 5^x \ln 5 + \frac{1}{x^2 + 1} \quad y = 5^x + \tan^{-1} x + C$
6. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{x}} \quad y = \sin^{-1} x - 2\sqrt{x} + C$
7. $\frac{dy}{dt} = 3t^2 \cos(t^3) \quad y = \sin(t^3) + C$
8. $\frac{dy}{dt} = (\cos t)\, e^{\sin t} \quad y = e^{\sin t} + C$
9. $\frac{du}{dx} = (\sec^2 x^5)(5x^4) \quad u = \tan(x^5) + C$
10. $\frac{dy}{du} = 4(\sin u)^3(\cos u) \quad y = (\sin u)^4 + C$

In Exercises 11–20, solve the initial value problem explicitly.

11. $\frac{dy}{dx} = 3\sin x$ and $y = 2$ when $x = 0$ $\quad y = -3\cos x + 5$
12. $\frac{dy}{dx} = 2e^x - \cos x$ and $y = 3$ when $x = 0$ $\quad y = 2e^x - \sin x + 1$
13. $\frac{du}{dx} = 7x^6 - 3x^2 + 5$ and $u = 1$ when $x = 1$ $\quad u = x^7 - x^3 + 5x - 4$
14. $\frac{dA}{dx} = 10x^9 + 5x^4 - 2x + 4$ and $A = 6$ when $x = 1$ $\quad A = x^{10} + x^5 - x^2 + 4x + 1$
15. $\frac{dy}{dx} = -\frac{1}{x^2} - \frac{3}{x^4} + 12$ and $y = 3$ when $x = 1$ $\quad y = x^{-1} + x^{-3} + 12x - 11 \; (x > 0)$
16. $\frac{dy}{dx} = 5\sec^2 x - \frac{3}{2}\sqrt{x}$ and $y = 7$ when $x = 0$ $\quad y = 5\tan x - x^{3/2} + 7 \; (0 < x < \pi/2)$
17. $\frac{dy}{dt} = \frac{1}{1 + t^2} + 2^t \ln 2$ and $y = 3$ when $t = 0$ $\quad y = \tan^{-1} t + 2^t + 2$
18. $\frac{dx}{dt} = \frac{1}{t} - \frac{1}{t^2} + 6$ and $x = 0$ when $t = 1$ $\quad x = \ln t + t^{-1} + 6t - 7 \; (t > 0)$
19. $\frac{dv}{dt} = 4\sec t \tan t + e^t + 6t$ and $v = 5$ when $t = 0$ $\quad v = 4\sec t + e^t + 3t^2 \; (-\pi/2 < t < \pi/2)$ (Note that $C = 0$.)
20. $\frac{ds}{dt} = t(3t - 2)$ and $s = 0$ when $t = 1$ $\quad s = t^3 - t^2$ (Note that $C = 0$.)

In Exercises 21–24, solve the initial value problem using the Fundamental Theorem. (Your answer will contain a definite integral.)

21. $\frac{dy}{dx} = \sin(x^2)$ and $y = 5$ when $x = 1$ $\quad y = \int_1^x \sin(t^2)\,dt + 5$
22. $\frac{du}{dx} = \sqrt{2 + \cos x}$ and $u = -3$ when $x = 0$ $\quad u = \int_0^x \sqrt{2 + \cos t}\,dt - 3$
23. $F'(x) = e^{\cos x}$ and $F(2) = 9$ $\quad F(x) = \int_2^x e^{\cos t}\,dt + 9$
24. $G'(s) = \sqrt[3]{\tan s}$ and $G(0) = 4$ $\quad G(s) = \int_0^s \sqrt[3]{\tan t}\,dt + 4$

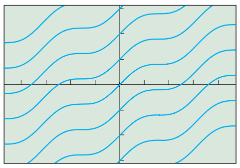
In Exercises 25–28, match the differential equation with the graph of a family of functions (a)–(d) that solve it. Use slope analysis, not your graphing calculator.

25. $\dfrac{dy}{dx} = (\sin x)^2$ Graph (b)

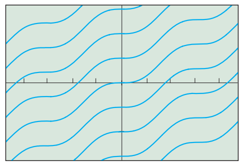
26. $\dfrac{dy}{dx} = (\sin x)^3$ Graph (c)

27. $\dfrac{dy}{dx} = (\cos x)^2$ Graph (a)

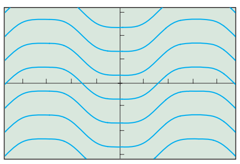
28. $\dfrac{dy}{dx} = (\cos x)^3$ Graph (d)



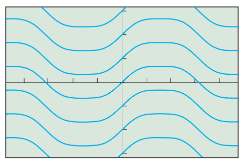
(a)



(b)

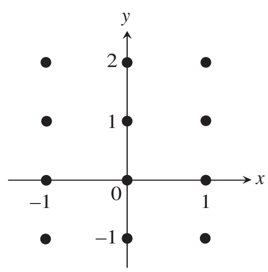


(c)



(d)

In Exercises 29–34, construct a slope field for the differential equation. In each case, copy the graph at the right and draw tiny segments through the twelve lattice points shown in the graph. Use slope analysis, not your graphing calculator.



29. $\dfrac{dy}{dx} = x$

30. $\dfrac{dy}{dx} = y$

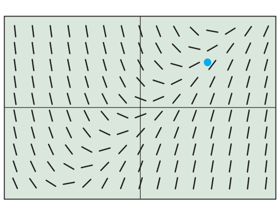
31. $\dfrac{dy}{dx} = 2x + y$

32. $\dfrac{dy}{dx} = 2x - y$

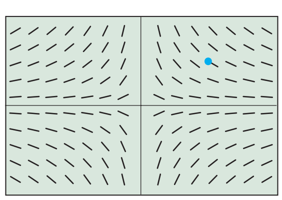
33. $\dfrac{dy}{dx} = x + 2y$

34. $\dfrac{dy}{dx} = x - 2y$

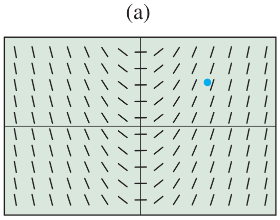
In Exercises 35–40, match the differential equation with the appropriate slope field. Then use the slope field to sketch the graph of the particular solution through the highlighted point (3, 2). (All slope fields are shown in the window [−6, 6] by [−4, 4].)



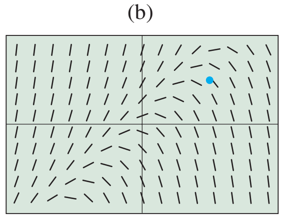
(a)



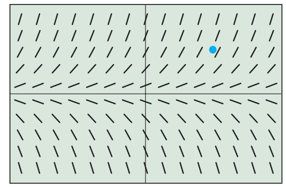
(b)



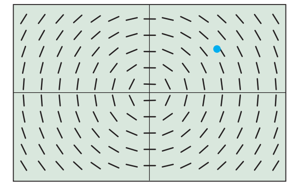
(c)



(d)



(e)



(f)

35. $\dfrac{dy}{dx} = x$

36. $\dfrac{dy}{dx} = y$

37. $\dfrac{dy}{dx} = x - y$

38. $\dfrac{dy}{dx} = y - x$

39. $\dfrac{dy}{dx} = -\dfrac{y}{x}$

40. $\dfrac{dy}{dx} = -\dfrac{x}{y}$

In Exercises 41–44, use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$.

41. $\dfrac{dy}{dx} = x - 1$ and $y = 2$ when $x = 1$ 2.03

42. $\dfrac{dy}{dx} = y - 1$ and $y = 3$ when $x = 1$ 3.662

43. $\dfrac{dy}{dx} = y - x$ and $y = 2$ when $x = 1$ 2.3

44. $\dfrac{dy}{dx} = 2x - y$ and $y = 0$ when $x = 1$ 0.6

In Exercises 45–48, use Euler's Method with increments of $\Delta x = -0.1$ to approximate the value of y when $x = 1.7$.

45. $\dfrac{dy}{dx} = 2 - x$ and $y = 1$ when $x = 2$ 0.97

46. $\dfrac{dy}{dx} = 1 + y$ and $y = 0$ when $x = 2$ −0.271

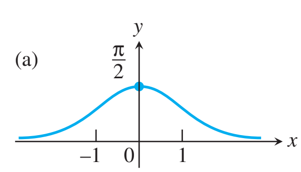
47. $\dfrac{dy}{dx} = x - y$ and $y = 2$ when $x = 2$ 2.031

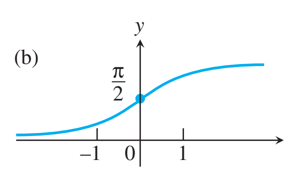
48. $\dfrac{dy}{dx} = x - 2y$ and $y = 1$ when $x = 2$ 1.032

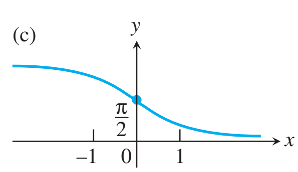
In Exercises 49 and 50, **(a)** determine which graph shows the solution of the initial value problem without actually solving the problem. **(b) Writing to Learn** Explain how you eliminated two of the possibilities.

49. $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}, \quad y(0) = \dfrac{\pi}{2}$

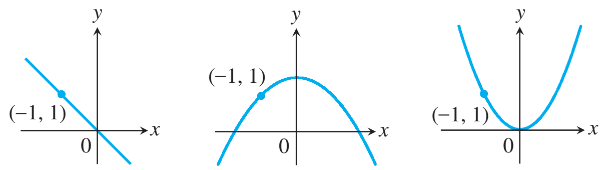
(a) Graph (b)
(b) The slope is always positive, so graphs (a) and (c) can be ruled out.





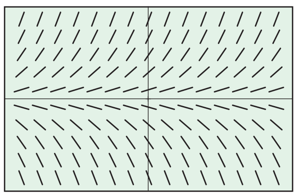


50. $\dfrac{dy}{dx} = -x, \quad y(-1) = 1$ See page 330.



(a) (b) (c)

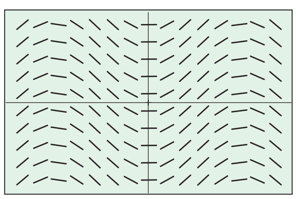
51. Writing to Learn Explain why $y = x^2$ could not be a solution to the differential equation with slope field shown below.



[–4.7, 4.7] by [–3.1, 3.1]

For one thing, there are positive slopes in the second quadrant of the slope field. The graph of $y = x^2$ has negative slopes in the second quadrant.

52. Writing to Learn Explain why $y = \sin x$ could not be a solution to the differential equation with slope field shown below.



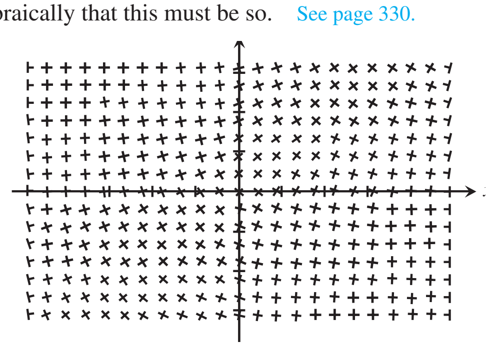
[–4.7, 4.7] by [–3.1, 3.1]

For one thing, the slope of $y = \sin x$ would be $+1$ at the origin, while the slope field shows a slope of zero at every point on the y-axis.

53. *Percentage Error* Let $y = f(x)$ be the solution to the initial value problem $dy/dx = 2x + 1$ such that $f(1) = 3$. Find the percentage error if Euler's Method with $\Delta x = 0.1$ is used to approximate $f(1.4)$. See page 330.

54. *Percentage Error* Let $y = f(x)$ be the solution to the initial value problem $dy/dx = 2x - 1$ such that $f(2) = 3$. Find the percentage error if Euler's Method with $\Delta x = -0.1$ is used to approximate $f(1.6)$. See page 330.

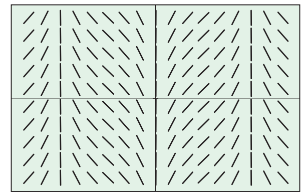
55. *Perpendicular Slope Fields* The figure below shows the slope fields for the differential equations $dy/dx = e^{(x-y)/2}$ and $dy/dx = -e^{(y-x)/2}$ superimposed on the same grid. It appears that the slope lines are perpendicular wherever they intersect. Prove algebraically that this must be so. See page 330.



56. *Perpendicular Slope Fields* If the slope fields for the differential equations $dy/dx = \sec x$ and $dy/dx = g(x)$ are perpendicular (as in Exercise 55), find $g(x)$.

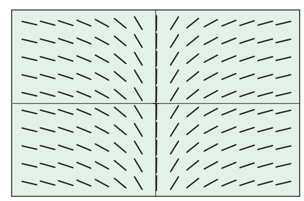
Since the slopes must be negative reciprocals, $g(x) = -\cos x$.

57. *Plowing Through a Slope Field* The slope field for the differential equation $dy/dx = \csc x$ is shown below. Find a function that will be *perpendicular* to every line it crosses in the slope field. (*Hint:* First find a differential equation that will produce a perpendicular slope field.) The perpendicular slope field would be produced by $dy/dx = -\sin x$, so $y = \cos x + C$ for any constant C.



[–4.7, 4.7] by [–3.1, 3.1]

58. *Plowing Through a Slope Field* The slope field for the differential equation $dy/dx = 1/x$ is shown below. Find a function that will be *perpendicular* to every line it crosses in the slope field. (*Hint:* First find a differential equation that will produce a perpendicular slope field.)



[–4.7, 4.7] by [–3.1, 3.1]

The perpendicular slope field would be produced by $dy/dx = -x$, so $y = -0.5x^2 + C$ for any constant C.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

59. True or False Any two solutions to the differential equation $dy/dx = 5$ are parallel lines. Justify your answer. True. They are all lines of the form $y = 5x + C$.

60. True or False If $f(x)$ is a solution to $dy/dx = 2x$, then $f^{-1}(x)$ is a solution to $dy/dx = 2y$. Justify your answer. See page 330.

61. Multiple Choice A slope field for the differential equation $dy/dx = 42 - y$ will show C

(A) a line with slope –1 and y-intercept 42.

(B) a vertical asymptote at $x = 42$.

(C) a horizontal asymptote at $y = 42$.

(D) a family of parabolas opening downward.

(E) a family of parabolas opening to the left.

62. Multiple Choice For which of the following differential equations will a slope field show nothing but negative slopes in the fourth quadrant? E

(A) $\dfrac{dy}{dx} = -\dfrac{x}{y}$ **(B)** $\dfrac{dy}{dx} = xy + 5$ **(C)** $\dfrac{dy}{dx} = xy^2 - 2$

(D) $\dfrac{dy}{dx} = \dfrac{x^3}{y^2}$ **(E)** $\dfrac{dy}{dx} = \dfrac{y}{x^2} - 3$

63. Multiple Choice If $dy/dx = 2xy$ and $y = 1$ when $x = 0$, then $y =$ B

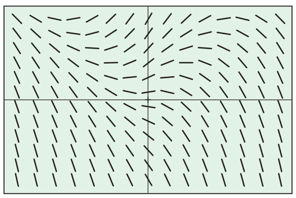
(A) y^{2x} **(B)** e^{x^2} **(C)** x^2y **(D)** $x^2y + 1$ **(E)** $\frac{x^2y^2}{2} + 1$

64. Multiple Choice Which of the following differential equations would produce the slope field shown below? A

(A) $\frac{dy}{dx} = y - |x|$ **(B)** $\frac{dy}{dx} = |y| - x$

(C) $\frac{dy}{dx} = |y - x|$ **(D)** $\frac{dy}{dx} = |y + x|$

(E) $\frac{dy}{dx} = |y| - |x|$



[–3, 3] by [–1.98, 1.98]

Explorations

65. *Solving Differential Equations* Let $\frac{dy}{dx} = x - \frac{1}{x^2}$.

(a) Find a solution to the differential equation in the interval (0,) that satisfies $y(1) = 2$. $y = \frac{x^2}{2} + \frac{1}{x} + \frac{1}{2}, x > 0$

(b) Find a solution to the differential equation in the interval $(-\infty, 0)$ that satisfies $y(-1) = 1$. $y = \frac{x^2}{2} + \frac{1}{x} + \frac{3}{2}, x < 0$

(c) Show that the following piecewise function is a solution to the differential equation for any values of C_1 and C_2.

$$y = \begin{cases} \frac{1}{x} + \frac{x^2}{2} + C_1, & x < 0 \\ \frac{1}{x} + \frac{x^2}{2} + C_2, & x > 0 \end{cases}$$

$$y' = \begin{cases} x - 1/x^2, & x < 0 \\ x - 1/x^2, & x > 0 \end{cases}$$

(d) Choose values for C_1 and C_2 so that the solution in part (c) agrees with the solutions in parts (a) and (b). $C_1 = \frac{3}{2}, C_2 = \frac{1}{2}$

(e) Choose values for C_1 and C_2 so that the solution in part (c) satisfies $y(2) = -1$ and $y(-2) = 2$. $C_1 = \frac{1}{2}, C_2 = -\frac{7}{2}$

66. *Solving Differential Equations* Let $\frac{dy}{dx} = \frac{1}{x}$.

(a) Show that $y = \ln x + C$ is a solution to the differential equation in the interval $(0, \infty)$. $\frac{d}{dx}(\ln x + C) = \frac{1}{x}$ for $x > 0$

(b) Show that $y = \ln(-x) + C$ is a solution to the differential equation in the interval $(-\infty, 0)$. $\frac{d}{dx}(\ln(-x) + C) = \frac{1}{x}$ for $x < 0$

(c) Writing to Learn Explain why $y = \ln|x| + C$ is a solution to the differential equation in the domain $(-\infty, 0) \cup (0, \infty)$. $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all x except 0.

(d) Show that the function

$$y = \begin{cases} \ln(-x) + C_1, & x < 0 \\ \ln x + C_2, & x > 0 \end{cases}$$

is a solution to the differential equation for any values of C_1 and C_2. $\frac{dy}{dx} = \frac{1}{x}$ for all x except 0.

Extending the Ideas

67. *Second-Order Differential Equations* Find the general solution to each of the following second-order differential equations by first finding dy/dx and then finding y. The general solution will have two unknown constants.

(a) $\frac{d^2y}{dx^2} = 12x + 4$ **(b)** $\frac{d^2y}{dx^2} = e^x + \sin x$ **(c)** $\frac{d^2y}{dx^2} = x^3 + x^{-3}$

68. *Second-Order Differential Equations* Find the specific solution to each of the following second-order initial value problems by first finding dy/dx and then finding y.

(a) $\frac{d^2y}{dx^2} = 24x^2 - 10$. When $x = 1$, $\frac{dy}{dx} = 3$ and $y = 5$. $y = 2x^4 - 5x^2 + 5x + 3$

(b) $\frac{d^2y}{dx^2} = \cos x - \sin x$. When $x = 0$, $\frac{dy}{dx} = 2$ and $y = 0$ $y = -\cos x + \sin x + x + 1$

(c) $\frac{d^2y}{dx^2} = e^x - x$. When $x = 0$, $\frac{dy}{dx} = 0$ and $y = 1$. $y = e^x - \frac{x^3}{6} - x + \frac{1}{6}$

69. *Differential Equation Potpourri* For each of the following differential equations, find at least one particular solution. You will need to call on past experience with functions you have differentiated. For a greater challenge, find the general solution.

(a) $y' = x$ **(b)** $y' = -x$ **(c)** $y' = y$

(d) $y' = -y$ **(e)** $y' = xy$

70. *Second-Order Potpourri* For each of the following second-order differential equations, find at least one particular solution. You will need to call on past experience with functions you have differentiated. For a significantly greater challenge, find the general solution (which will involve two unknown constants).

(a) $y'' = x$ **(b)** $y'' = -x$ **(c)** $y'' = -\sin x$

(d) $y'' = y$ **(e)** $y'' = -y$

Answers:

50. (a) Graph (b)

(b) The solution should have positive slope when x is negative, zero slope when x is zero and negative slope when x is positive since slope $= dy/dx = -x$. Graphs (a) and (c) don't show this slope pattern.

53. Euler's Method gives an estimate $f(1.4) \approx 4.32$. The solution to the initial value problem is $f(x) = x^2 + x + 1$, from which we get $f(1.4) = 4.36$. The percentage error is thus $(4.36 - 4.32)/4.36 = 0.9\%$.

54. Euler's Method gives an estimate $f(1.6) \approx 1.92$. The solution to the initial value problem is $f(x) = x^2 - x + 1$, from which we get $f(1.6) = 1.96$. The percentage error is thus $(1.96 - 1.92)/1.96 = 2\%$.

55. At every point (x, y), $(e^{(x-y)/2})(-e^{(y-x)/2}) = -e^{(x-y)/2+(y-x)/2} = -e^0 = -1$, so the slopes are negative reciprocals. The slope lines are therefore perpendicular.

60. False. For example, $f(x) = x^2$ is a solution of $dy/dx = 2x$, but $f^{-1}(x) = \sqrt{x}$ is not a solution of $dy/dx = 2y$.

67. (a) $y = 2x^3 + 2x^2 + C_1x + C_2$ **(b)** $y = e^x - \sin x + C_1x + C_2$

(c) $y = \frac{x^5}{20} + \frac{x^{-1}}{2} + C_1x + C_2$

69. (a) $y = \frac{x^2}{2} + C$ **(b)** $y = -\frac{x^2}{2} + C$ **(c)** $y = Ce^x$

(d) $y = Ce^{-x}$ **(e)** $y = Ce^{x^2/2}$

70. (a) $y = \frac{x^3}{6} + C_1x + C_2$ **(b)** $y = -\frac{x^3}{6} + C_1x + C_2$

(c) $y = \sin x + C_1x + C_2$ **(d)** $y = C_1e^x + C_2e^{-x}$

(e) $y = C_1 \sin x + C_2 \cos x$

6.2 Antidifferentiation by Substitution

What you'll learn about

- Indefinite Integrals
- Leibniz Notation and Antiderivatives
- Substitution in Indefinite Integrals
- Substitution in Definite Integrals

. . . and why

Antidifferentiation techniques were historically crucial for applying the results of calculus.

Indefinite Integrals

If $y = f(x)$ we can denote the derivative of f by either dy/dx or $f'(x)$. What can we use to denote the *antiderivative* of f? We have seen that the general solution to the differential equation $dy/dx = f(x)$ actually consists of an infinite family of functions of the form $F(x) + C$, where $F'(x) = f(x)$. Both the name for this family of functions and the symbol we use to denote it are closely related to the definite integral because of the Fundamental Theorem of Calculus.

DEFINITION Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x)dx$.

If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration.**

As in Chapter 5, the symbol $\int$ is an **integral sign,** the function f is the **integrand** of the integral, and x is the **variable of integration.**

Notice that an indefinite integral is not at all like a definite integral, despite the similarities in notation and name. A definite integral is a *number*, the limit of a sequence of Riemann sums. An indefinite integral is a *family of functions* having a common derivative. If the Fundamental Theorem of Calculus had not provided such a dramatic link between antiderivatives and integration, we would surely be using a different name and symbol for the general antiderivative today.

EXAMPLE 1 Evaluating an Indefinite Integral

Evaluate $\int (x^2 - \sin x)\, dx$.

SOLUTION

Evaluating this definite integral is just like solving the differential equation $dy/dx = x^2 - \sin x$. Our past experience with derivatives leads us to conclude that

$$\int (x^2 - \sin x)\,dx = \frac{x^3}{3} + \cos x + C$$

(as you can check by differentiating).

Now try Exercise 3.

You have actually been finding antiderivatives since Section 5.3, so Example 1 should hardly have seemed new. Indeed, each derivative formula in Chapter 3 could be turned around to yield a corresponding indefinite integral formula. We list some of the most useful such indefinite integral formulas below. Be sure to familiarize yourself with these before moving on to the next section, in which function composition becomes an issue. (Incidentally, it is in anticipation of the next section that we give some of these formulas in terms of the variable u rather than x.)

Properties of Indefinite Integrals

$$\int k f(x)dx = k\int f(x)\,dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Power Formulas

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \text{ when } n \neq -1$$

$$\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$$
(see Example 2)

Trigonometric Formulas

$$\int \cos u\,du = \sin u + C \qquad \int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C \qquad \int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C \qquad \int \csc u \cot u\,du = -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u\,du = e^u + C \qquad \int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \ln u\,du = u\ln u - u + C \quad \text{(See Example 2)}$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}\,du = \frac{u\ln u - u}{\ln a} + C$$

A Note on Absolute Value

Since the indefinite integral does not specify a domain, you should always use the absolute value when finding $\int 1/u\,du$. The function $\ln u + C$ is only defined on positive u-intervals, while the function $\ln|u| + C$ is defined on both the positive *and* negative intervals in the domain of $1/u$ (see Example 2).

EXAMPLE 2 Verifying Antiderivative Formulas

Verify the antiderivative formulas:

(a) $\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$ **(b)** $\int \ln u\,du = u\ln u - u + C$

SOLUTION

We can verify antiderivative formulas by differentiating.

(a) For $u > 0$, we have $\frac{d}{du}(\ln|u| + C) = \frac{d}{du}(\ln u + C) = \frac{1}{u} + 0 = \frac{1}{u}$.

For $u < 0$, we have $\frac{d}{du}(\ln|u| + C) = \frac{d}{du}(\ln(-u) + C) = \frac{1}{-u}(-1) + 0 = \frac{1}{u}$.

Since $\frac{d}{du}(\ln|u| + C) = \frac{1}{u}$ in either case, $\ln|u| + C$ is the general antiderivative of the function $\frac{1}{u}$ on its entire domain.

(b) $\frac{d}{du}(u\ln u - u + C) = 1 \cdot \ln u + u\left(\frac{1}{u}\right) - 1 + 0 = \ln u + 1 - 1 = \ln u$.

Now try Exercise 11.

Leibniz Notation and Antiderivatives

The appearance of the differential "dx" in the definite integral $\int_a^b f(x)dx$ is easily explained by the fact that it is the limit of a Riemann sum of the form $\sum_{k=1}^{n} f(x_k) \cdot \Delta x$ (see Section 5.2). The same "dx" almost seems unnecessary when we use the indefinite integral $\int f(x)dx$ to represent the general antiderivative of f, but in fact it is quite useful for *dealing with the effects of the Chain Rule* when function composition is involved. Exploration 1 will show you why this is an important consideration.

EXPLORATION 1 **Are $\int f(u)\,du$ and $\int f(u)\,dx$ the Same Thing?**

Let $u = x^2$ and let $f(u) = u^3$.

1. Find $\int f(u)\,du$ as a function of u.
2. Use your answer to question 1 to write $\int f(u)\,du$ as a function of x.
3. Show that $f(u) = x^6$ and find $\int f(u)\,dx$ as a function of x.
4. Are the answers to questions 2 and 3 the same?

Exploration 1 shows that the notation $\int f(u)$ is not sufficient to describe an antiderivative when u is a function of another variable. Just as du/du is different from du/dx when differentiating, $\int f(u)\,du$ is different from $\int f(u)\,dx$ when antidifferentiating. We will use this fact to our advantage in the next section, where the importance of "dx" or "du" in the integral expression will become even more apparent.

EXAMPLE 3 Paying Attention to the Differential

Let $f(x) = x^3 + 1$ and let $u = x^2$. Find each of the following antiderivatives in terms of x:

(a) $\displaystyle\int f(x)\,dx$ **(b)** $\displaystyle\int f(u)\,du$ **(c)** $\displaystyle\int f(u)\,dx$

SOLUTION

(a) $$\int f(x)\,dx = \int (x^3 + 1)\,dx = \frac{x^4}{4} + x + C$$

(b) $$\int f(u)\,du = \int (u^3 + 1)\,du = \frac{u^4}{4} + u + C = \frac{(x^2)^4}{4} + x^2 + C = \frac{x^8}{4} + x^2 + C$$

(c) $$\int f(u)\,dx = \int (u^3 + 1)\,dx = \int ((x^2)^3 + 1)\,dx = \int (x^6 + 1)\,dx = \frac{x^7}{7} + x + C$$

Now try Exercise 15.

Substitution in Indefinite Integrals

A change of variables can often turn an unfamiliar integral into one that we can evaluate. The important point to remember is that it is *not sufficient* to change an integral of the form $\int f(x)\,dx$ into an integral of the form $\int g(u)\,dx$. The differential matters. A complete substitution changes the integral $\int f(x)\,dx$ into an integral of the form $\int g(u)\,du$.

EXAMPLE 4 Using Substitution

Evaluate $\int \sin x\, e^{\cos x}\,dx$.

continued

SOLUTION

Let $u = \cos x$. Then $du/dx = -\sin x$, from which we conclude that $du = -\sin x\, dx$. We rewrite the integral and proceed as follows:

$$\int \sin x\, e^{\cos x}\, dx = -\int (-\sin x)e^{\cos x}\, dx$$

$$= -\int e^{\cos x} \cdot (-\sin x)dx$$

$$= -\int e^u\, du$$ Substitute u for $\cos x$ and du for $-\sin x\, dx$.

$$= -e^u + C$$

$$= -e^{\cos x} + C$$ Re-substitute $\cos x$ for u after antidifferentiating.

Now try Exercise 19.

If you differentiate $-e^{\cos x} + C$, you will find that a factor of $-\sin x$ appears when you apply the Chain Rule. The technique of *antidifferentiation by substitution* reverses that effect by absorbing the $-\sin x$ into the differential du when you change $\int \sin x\, e^{\cos x}\, dx$ into $-\int e^u\, du$. That is why a "u-substitution" always involves a "du-substitution" to convert the integral into a form ready for antidifferentiation.

EXAMPLE 5 Using Substitution

Evaluate $\int x^2 \sqrt{5 + 2x^3}\, dx$.

SOLUTION

This integral invites the substitution $u = 5 + 2x^3$, $du = 6x^2\, dx$.

$$\int x^2 \sqrt{5 + 2x^3}\, dx. = \int (5 + 2x^3)^{1/2} \cdot x^2\, dx$$

$$= \frac{1}{6}\int (5 + 2x^3)^{1/2} \cdot 6x^2\, dx$$ Set up the substitution with a factor of 6.

$$= \frac{1}{6}\int u^{1/2}\, du$$ Substitute u for $5 + 2x^3$ and du for $6x^2\, dx$.

$$= \frac{1}{6}\left(\frac{2}{3}\right)u^{3/2} + C$$

$$= \frac{1}{9}(5 + 2x^3)^{3/2} + C$$ Re-substitute after antidifferentiating.

Now try Exercise 27.

EXAMPLE 6 Using Substitution

Evaluate $\int \cot 7x\, dx$.

SOLUTION

We do not recall a function whose derivative is $\cot 7x$, but a basic trigonometric identity changes the integrand into a form that invites the substitution $u = \sin 7x$, $du = 7\cos 7x\, dx$. We rewrite the integrand as shown on the next page.

$$\int \cot 7x\,dx = \int \frac{\cos 7x}{\sin 7x}\,dx$$ Trigonometric identity

$$= \frac{1}{7}\int \frac{7\cos 7x\,dx}{\sin x}$$ Note that $du = 7\cos 7x\,dx$ when $u = \sin 7x$

We multiply by $\frac{1}{7} \cdot 7$, or 1.

$$= \frac{1}{7}\int \frac{du}{u}$$ Substitute u for 7 sin x and du for 7 cos 7x dx.

$$= \frac{1}{7}\ln|u| + C$$ Notice the absolute value!

$$= \frac{1}{7}\ln|\sin 7x| + C$$ Re-substitute sin 7x for u after antidifferentiating.

Now try Exercise 29.

EXAMPLE 7 Setting Up a Substitution with a Trigonometric Identity

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a) $\int \frac{dx}{\cos^2 2x}$ **(b)** $\int \cot^2 3x\,dx$ **(c)** $\int \cos^3 x\,dx$

SOLUTION

(a) $$\int \frac{dx}{\cos^2 2x} = \int \sec^2 2x\,dx$$

$$= \frac{1}{2}\int \sec^2 2x \cdot 2dx$$

$$= \frac{1}{2}\int \sec^2 u\,du$$ Let $u = 2x$ and $du = 2\,dx$.

$$= \frac{1}{2}\tan u + C$$

$$= \frac{1}{2}\tan 2x + C$$ Re-substitute after antidifferentiating.

(b) $$\int \cot^2 3x\,dx = \int (\csc^2 3x - 1)\,dx$$

$$= \frac{1}{3}\int (\csc^2 3x - 1) \cdot 3\,dx$$

$$= \frac{1}{3}\int (\csc^2 u - 1) \cdot du$$ Let $u = 3x$ and $du = 3dx$.

$$= \frac{1}{3}\int (-\cot u - u) + C$$

$$= \frac{1}{3}(-\cot 3x - 3x) + C$$

$$= -\frac{1}{3}\cot 3x - x + C$$ Re-substitute after antidifferentiation.

continued

(c) $\int \cos^3 x\, dx = \int (\cos^2 x) \cos x\, dx$

$= \int (1 - \sin^2 x) \cos x\, dx$

$= \int (1 - u^2)\, du$ Let $u = \sin x$ and $du = \cos x\, dx$.

$= u - \frac{u^3}{3} + C$

$= \sin x - \frac{\sin^3 x}{3} + C$ Re-substitute after antidifferentiating.

Now try Exercise 47.

Substitution in Definite Integrals

Antiderivatives play an important role when we evaluate a definite integral by the Fundamental Theorem of Calculus, and so, consequently, does substitution. In fact, if we make full use of our substitution of variables and change the interval of integration to match the u-substitution in the integrand, we can avoid the "resubstitution" step in the previous four examples.

EXAMPLE 8 Evaluating a Definite Integral by Substitution

Evaluate $\int_0^{\pi/3} \tan x \sec^2 x\, dx$.

SOLUTION

Let $u = \tan x$ and $du = \sec^2 x\, dx$.

Note also that $u(0) = \tan 0 = 0$ and $u(\pi/3) = \tan(\pi/3) = \sqrt{3}$.

So

$$\int_0^{\pi/3} \tan x \sec^2 x\, dx = \int_0^{\sqrt{3}} u\, du$$ Substitute u-interval for x-interval.

$$= \left. \frac{u^2}{2} \right|_0^{\sqrt{3}}$$

$$= \frac{3}{2} - 0 = \frac{3}{2}$$

Now try Exercise 55.

EXAMPLE 9 That Absolute Value Again

Evaluate $\int_0^1 \frac{x}{x^2 - 4}\, dx$.

SOLUTION

Let $u = x^2 - 4$ and $du = 2x\, dx$. Then $u(0) = 0^2 - 4 = -4$ and $u(1) = 1^2 - 4 = -3$.

continued

So

$$\int_0^1 \frac{x}{x^2-4}\,dx = \frac{1}{2}\int_0^1 \frac{2x\ dx}{x^2-4}$$

$$= \frac{1}{2}\int_{-4}^{-3} \frac{du}{u} \qquad \text{Substitute } u\text{-interval for } x\text{-interval.}$$

$$= \frac{1}{2}\ln|u|\Big|_{-4}^{-3}$$

$$= \frac{1}{2}(\ln 3 - \ln 4) = \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

Notice that ln u would not have existed over the interval of integration $[-4, -3]$. The absolute value in the antiderivative is important. ***Now try Exercise 63.***

Finally, consider this historical note. The technique of u-substitution derived its importance from the fact that it was a powerful tool for antidifferentiation. Antidifferentiation derived its importance from the Fundamental Theorem, which established it as the way to evaluate definite integrals. Definite integrals derived their importance from real-world applications. While the applications are no less important today, the fact that the definite integrals can be easily evaluated by technology has made the world less reliant on antidifferentiation, and hence less reliant on u-substitution. Consequently, you have seen in this book only a sampling of the substitution tricks calculus students would have routinely studied in the past. You may see more of them in a differential equations course.

Quick Review 6.2 *(For help, go to Sections 3.6 and 3.9.)*

In Exercises 1 and 2, evaluate the definite integral.

1. $\int_0^2 x^4\,dx$ $32/5$

2. $\int_1^5 \sqrt{x-1}\,dx$ $16/3$

In Exercises 3–10, find dy/dx.

3. $y = \int_2^x 3^t\,dt$ 3^x

4. $y = \int_0^x 3^t\,dt$ 3^x

5. $y = (x^3 - 2x^2 + 3)^4$ $4(x^3 - 2x^2 + 3)^3(3x^2 - 4x)$

6. $y = \sin^2(4x - 5)$ $8\sin(4x-5)\cos(4x-5)$

7. $y = \ln\cos x$ $-\tan x$

8. $y = \ln\sin x$ $\cot x$

9. $y = \ln(\sec x + \tan x)$ $\sec x$

10. $y = \ln(\csc x + \cot x)$ $-\csc x$

Section 6.2 Exercises

In Exercises 1–6, find the indefinite integral.

1. $\int (\cos x - 3x^2)\,dx$ $\sin x - x^3 + C$

2. $\int x^{-2}\,dx$ $-x^{-1} + C$

3. $\int \left(t^2 - \frac{1}{t^2}\right)dt$ $t^3/3 + t^{-1} + C$

4. $\int \frac{dt}{t^2+1}$ $\tan^{-1} t + C$

5. $\int (3x^4 - 2x^{-3} + \sec^2 x)\,dx$ $(3/5)x^5 + x^{-2} + \tan x + C$

6. $\int (2e^x + \sec x\tan x - \sqrt{x})\,dx$ $2e^x + \sec x - (2/3)x^{3/2} + C$

In Exercises 7–12, use differentiation to verify the antiderivative formula.

7. $\int \csc^2 u\,du = -\cot u + C$ $(-\cot u + C)' = -(-\csc^2 u) = \csc^2 u$

8. $\int \csc u \cot u = -\csc u + C$ $(-\csc u + C)' = -(-\csc u\cot u) = \csc u\cot u$

9. $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$ See page 340.

10. $\int 5^x\,dx = \frac{1}{\ln 5}5^x + C$ See page 340.

11. $\int \frac{1}{1+u^2}\,du = \tan^{-1} u + C$ See page 340.

12. $\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$ See page 340.

In Exercises 13–16, verify that $\int f(u)\,du \neq \int f(u)\,dx$

13. $f(u) = \sqrt{u}$ and $u = x^2\ (x > 0)$ See page 340.

14. $f(u) = u^2$ and $u = x^5$ See page 340.

15. $f(u) = e^u$ and $u = 7x$ See page 340.

16. $f(u) = \sin u$ and $u = 4x$ See page 340.

In Exercises 17–24, use the indicated substitution to evaluate the integral. Confirm your answer by differentiation.

17. $\int \sin 3x\,dx, \quad u = 3x$ $-\frac{1}{3}\cos 3x + C$

18. $\int x\cos(2x^2)\,dx, \quad u = 2x^2$ $\frac{1}{4}\sin(2x^2) + C$

19. $\int \sec 2x \tan 2x\,dx, \quad u = 2x$ $\frac{1}{2}\sec 2x + C$

20. $\int 28(7x-2)^3\,dx, \quad u = 7x - 2$ $(7x-2)^4 + C$

21. $\int \frac{dx}{x^2+9}, \quad u = \frac{x}{3}$ $(1/3)\tan^{-1}(x/3) + C$

22. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$ $-6\sqrt{1-r^3} + C$

23. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$ $\frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$

24. $\int 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)\,dy, \quad u = y^4 + 4y^2 + 1$ $\frac{2}{3}(y^4 + 4y^2 + 1)^3 + C$

In Exercises 25–46, use substitution to evaluate the integral.

25. $\int \frac{dx}{(1-x)^2}$ $\frac{1}{1-x} + C$

26. $\int \sec^2(x+2)\,dx$ $\tan(x+2) + C$

27. $\int \sqrt{\tan x}\sec^2 x\,dx$ $\frac{2}{3}(\tan x)^{3/2} + C$

28. $\int \sec\left(\theta + \frac{\pi}{2}\right)\tan\left(\theta + \frac{\pi}{2}\right)d\theta$ $\sec\left(\theta + \frac{\pi}{2}\right) + C$

29. $\int \tan(4x+2)\,dx$

30. $\int 3(\sin x)^{-2}\,dx$ $-3\cot x + C$

31. $\int \cos(3z+4)\,dz$ $\frac{1}{3}\sin(3z+4) + C$

32. $\int \sqrt{\cot x}\csc^2 x\,dx$ $-\frac{2}{3}(\cot x)^{3/2} + C$

33. $\int \frac{\ln^6 x}{x}\,dx$ $\frac{1}{7}(\ln x)^7 + C$

34. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$ $\frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C$

35. $\int s^{1/3}\cos(s^{4/3} - 8)\,ds$ $\frac{3}{4}\sin(s^{4/3} - 8) + C$

36. $\int \frac{dx}{\sin^2 3x}$ $-\frac{1}{3}\cot(3x) + C$

37. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$ $(1/2)\sec(2t+1) + C$

38. $\int \frac{6\cos t}{(2+\sin t)^2}\,dt$ $-\frac{6}{2+\sin t} + C$

39. $\int \frac{dx}{x\ln x}$ $\ln(\ln x) + C$

40. $\int \tan^2 x\sec^2 x\,dx$ $(1/3)\tan^3 x + C$

41. $\int \frac{x\,dx}{x^2+1}$ $(1/2)\ln(x^2+1) + C$

42. $\int \frac{40\,dx}{x^2+25}$ $8\tan^{-1}\left(\frac{x}{5}\right) + C$

29. $-(1/4)\ln|\cos(4x+2)| + C$ or $(1/4)\ln|\sec(4x+2)| + C$

43. $\int \frac{dx}{\cot 3x}$

43. $\frac{1}{3}\ln|\sec(3x)| + C = -\frac{1}{3}\ln|\cos(3x)| + C$

44. $\int \frac{dx}{\sqrt{5x+8}}$ $\frac{2}{5}\sqrt{5x+8} + C$

45. $\int \sec x\,dx$ (*Hint:* Multiply the integrand by

$$\frac{\sec x + \tan x}{\sec x + \tan x}$$ $\ln|\sec x + \tan x| + C$

and then use a substitution to integrate the result.)

46. $\int \csc x\,dx$ (*Hint:* Multiply the integrand by

$$\frac{\csc x + \cot x}{\csc x + \cot x}$$ $-\ln|\csc x + \cot x| + C$

and then use a substitution to integrate the result.)

In Exercises 47–52, use the given trigonometric identity to set up a u-substitution and then evaluate the indefinite integral.

47. $\int \sin^3 2x\,dx, \quad \sin^2 2x = 1 - \cos^2 2x$ $\frac{\cos^3 x}{3} - \frac{\cos x}{2} + C$

48. $\int \sec^4 x\,dx, \quad \sec^2 x = 1 + \tan^2 x$ $\tan x + \frac{\tan^3 x}{3} + C$

49. $\int 2\sin^2 x\,dx, \quad \cos 2x = 2\sin^2 x - 1$ $x + \frac{\sin 2x}{2} + C$

50. $\int 4\cos^2 x\,dx, \quad \cos 2x = 1 - 2\cos^2 x$ $2x - \sin 2x + C$

51. $\int \tan^4 x\,dx, \quad \tan^2 x = \sec^2 x - 1$ $\frac{1}{3}\tan^3 x - \tan x + x + C$

52. $\int (\cos^4 x - \sin^4 x)dx, \quad \cos 2x = \cos^2 x - \sin^2 x$ $\frac{1}{2}\sin 2x + C$

In Exercises 53–66, make a u-substitution and integrate from $u(a)$ to $u(b)$.

53. $\int_0^3 \sqrt{y+1}\,dy$ 14/3

54. $\int_0^1 r\sqrt{1-r^2}\,dr$ 1/3

55. $\int_{-\pi/4}^{0} \tan x\sec^2 x\,dx$ −1/2

56. $\int_{-1}^{1} \frac{5r}{(4+r^2)^2}\,dr$ 0

57. $\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\,d\theta$ 10/3

58. $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$ 0

59. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$ $2\sqrt{3}$

60. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$ 3/4

61. $\int_0^7 \frac{dx}{x+2}$ 1.504

62. $\int_2^5 \frac{dx}{2x-3}$ 0.973

63. $\int_1^2 \frac{dt}{t-3}$ −0.693

64. $\int_{\pi/4}^{3\pi/4} \cot x\,dx$ 0

65. $\int_{-1}^{3} \frac{x\,dx}{x^2+1}$ 0.805

66. $\int_0^2 \frac{e^x\,dx}{3+e^x}$ 0.954

72. True. Using the substitution $u = f(x)$, $du = f'(x)dx$, we have

$$\int_a^b \frac{f'(x)dx}{f(x)} = \int_{f(a)}^{f(b)} \frac{du}{u} = \ln u\Big|_{f(a)}^{f(b)} = \ln(f(b)) - \ln(f(a)) = \ln\left(\frac{f(b)}{f(a)}\right).$$

Two Routes to the Integral In Exercises 67 and 68, make a substitution $u = \cdots$ (an expression in x), $du = \cdots$. Then

(a) integrate with respect to u from $u(a)$ to $u(b)$.

(b) find an antiderivative with respect to u, replace u by the expression in x, then evaluate from a to b.

67. $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ **(a)** $\frac{1}{2}\sqrt{10} - \frac{3}{2} \approx 0.081$ **(b)** $\frac{1}{2}\sqrt{10} - \frac{3}{2} \approx 0.081$

68. $\displaystyle\int_{\pi/6}^{\pi/3} (1 - \cos 3x)\sin 3x\,dx$ **(a)** 1/2 **(b)** 1/2

69. Show that

$$y = \ln\left|\frac{\cos 3}{\cos x}\right| + 5$$

Note that $dy/dx = \tan x$ and $y(3) = 5$.

is the solution to the initial value problem

$$\frac{dy}{dx} = \tan x, \quad f(3) = 5.$$

(See the discussion following Example 4, Section 5.4.)

70. Show that

$$y = \ln\left|\frac{\sin x}{\sin 2}\right| + 6$$

Note that $dy/dx = \cot x$ and $y(2) = 6$.

is the solution to the initial value problem

$$\frac{dy}{dx} = \cot x, \quad f(2) = 6.$$

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

71. True or False By u-substitution, $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx = \int_0^{\pi/4} u^3\,du$. Justify your answer. See below.

72. True or False If f is positive and differentiable on $[a, b]$, then $\displaystyle\int_a^b \frac{f'(x)dx}{f(x)} = \ln\left(\frac{f(b)}{f(a)}\right)$. Justify your answer. See above.

73. Multiple Choice $\int \tan x\,dx =$ D

(A) $\frac{\tan^2 x}{2} + C$

(B) $\ln|\cot x| + C$

(C) $\ln|\cos x| + C$

(D) $-\ln|\cos x| + C$

(E) $-\ln|\cot x| + C$

74. Multiple Choice $\int_0^2 e^{2x}\,dx =$ E

(A) $\frac{e^4}{2}$ **(B)** $e^4 - 1$ **(C)** $e^4 - 2$ **(D)** $2e^4 - 2$ **(E)** $\frac{e^4 - 1}{2}$

75. Multiple Choice If $\int_3^5 f(x - a)\,dx = 7$ where a is a constant, then $\int_{3-a}^{5-a} f(x)\,dx =$ B

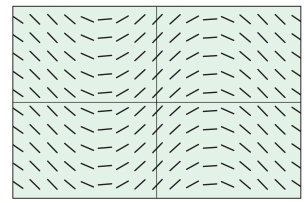
(A) $7 + a$ **(B)** 7 **(C)** $7 - a$ **(D)** $a - 7$ **(E)** -7

71. False. The interval of integration should change from $[0, \pi/4]$ to $[0, 1]$, resulting in a different numerical answer.

76. Multiple Choice If the differential equation $dy/dx = f(x)$ leads to the slope field shown below, which of the following could be $\int f(x)\,dx$? A

(A) $\sin x + C$ **(B)** $\cos x + C$ **(C)** $-\sin x + C$

(D) $-\cos x + C$ **(E)** $\frac{\sin^2 x}{2} + C$



Explorations

77. *Constant of Integration* Consider the integral

$$\int \sqrt{x+1}\,dx.$$

(a) Show that $\displaystyle\int \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{3/2} + C.$

(b) Writing to Learn Explain why See page 340.

$$y_1 = \int_0^x \sqrt{t+1}\,dt \quad \text{and} \quad y_2 = \int_3^x \sqrt{t+1}\,dt$$

are antiderivatives of $\sqrt{x+1}$.

(c) Use a table of values for $y_1 - y_2$ to find the value of C for which $y_1 = y_2 + C$. See page 340.

(d) Writing to Learn Give a convincing argument that See page 340.

$$C = \int_0^3 \sqrt{x+1}\,dx.$$

78. Group Activity *Making Connections* Suppose that

$$\int f(x)\,dx = F(x) + C.$$

(a) Explain how you can use the derivative of $F(x) + C$ to confirm the integration is correct.

(b) Explain how you can use a slope field of f and the graph of $y = F(x)$ to support your evaluation of the integral.

(c) Explain how you can use the graphs of $y_1 = F(x)$ and $y_2 = \int_0^x f(t)\,dt$ to support your evaluation of the integral.

(d) Explain how you can use a table of values for $y_1 - y_2$, y_1 and y_2 defined as in part (c), to support your evaluation of the integral.

(e) Explain how you can use graphs of f and NDER of $F(x)$ to support your evaluation of the integral.

(f) Illustrate parts (a)–(e) for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

79. ***Different Solutions?*** Consider the integral $\int 2\sin x\cos x\,dx$.

(a) Evaluate the integral using the substitution $u = \sin x$.

(b) Evaluate the integral using the substitution $u = \cos x$.

(c) Writing to Learn Explain why the different-looking answers in parts (a) and (b) are actually equivalent.

80. ***Different Solutions?*** Consider the integral $\int 2\sec^2 x\tan x\,dx$.

(a) Evaluate the integral using the substitution $u = \tan x$.

(b) Evaluate the integral using the substitution $u = \sec x$.

(c) Writing to Learn Explain why the different-looking answers in parts (a) and (b) are actually equivalent.

Extending the Ideas

81. ***Trigonometric Substitution*** Suppose $u = \sin^{-1} x$. Then $\cos u > 0$.

(a) Use the substitution $x = \sin u$, $dx = \cos u\,du$ to show that

$$\int \frac{dx}{\sqrt{1-x^2}} = \int 1\,du.$$

(b) Evaluate $\int 1\,du$ to show that $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$.

82. ***Trigonometric Substitution*** Suppose $u = \tan^{-1} x$.

(a) Use the substitution $x = \tan u$, $dx = \sec^2 u\,du$ to show that

$$\int \frac{dx}{1+x^2} = \int 1\,du.$$

(b) Evaluate $\int 1\,du$ to show that $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$.

83. ***Trigonometric Substitution*** Suppose $\sqrt{x} = \sin y$.

(a) Use the substitution $x = \sin^2 y$, $dx = 2\sin y\cos y\,dy$ to show that

$$\int_0^{1/2} \frac{\sqrt{x}\,dx}{\sqrt{1-x}} = \int_0^{\pi/4} 2\sin^2 y\,dy.$$

(b) Use the identity given in Exercise 49 to evaluate the definite integral without a calculator.

84. ***Trigonometric Substitution*** Suppose $u = \tan^{-1} x$.

(a) Use the substitution $x = \tan u$, $dx = \sec^2 u\,du$ to show that

$$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}} = \int_0^{\pi/3} \sec u\,du.$$

(b) Use the hint in Exercise 45 to evaluate the definite integral without a calculator.

Answers:

9. $\left(\frac{1}{2}e^{2x} + C\right)' = \frac{1}{2}e^{2x}\cdot 2 = e^{2x}$

10. $\left(\frac{1}{\ln 5}5^x + C\right)' = \frac{1}{\ln 5}5^x \cdot \ln 5 = 5^x$

11. $(\tan^{-1} u + C)' = \frac{1}{1+u^2}$ **12.** $(\sin^{-1} u + C)' = \frac{1}{\sqrt{1-u^2}}$

13. $\int f(u)\,du = \int \sqrt{u}\,du = (2/3)u^{3/2} + C = (2/3)x^3 + C$

$\int f(u)\,dx = \int \sqrt{u}\,dx = \int \sqrt{x^2}\,dx = \int x\,dx = (1/2)x^2 + C$

14. $\int f(u)\,du = \int u^2\,du = (1/3)u^3 + C = (1/3)x^{15} + C$

$\int f(u)\,dx = \int u^2\,dx = \int x^{10}\,dx = (1/11)x^{11} + C$

15. $\int f(u)\,du = \int e^u\,du = e^u + C = e^{7x} + C$

$\int f(u)\,dx = \int e^u\,du = \int e^{7x}\,dx = (1/7)e^{7x} + C$

16. $\int f(u)du = \int \sin u\,du = -\cos u + C = -\cos 4x + C$

$\int f(u)\,dx = \int \sin u\,dx = \int \sin 4x\,dx = -(1/4)\cos 4x + C$

77. (a) $\frac{d}{dx}\left(\frac{2}{3}(x+1)^{3/2} + C\right) = \sqrt{x+1}$

(b) Because $dy_1/dx = \sqrt{x+1}$ and $dy_2/dx = \sqrt{x+1}$

(c) $4\frac{2}{3}$ **(d)** $C = y_1 - y_2$

$= \int_0^x \sqrt{x+1}\,dx - \int_3^x \sqrt{x+1}\,dx$

$= \int_0^x \sqrt{x+1}\,dx + \int_x^3 \sqrt{x+1}\,dx$

$= \int_0^3 \sqrt{x+1}\,dx$

79. (a) $\int 2\sin x\cos x\,dx = \int 2u\,du = u^2 + C = \sin^2 x + C$

(b) $\int 2\sin x\cos x\,dx = -\int 2u\,du = -u^2 + C = -\cos^2 x + C$

(c) Since $\sin^2 x - (-\cos^2 x) = 1$, the two answers differ by a constant (accounted for in the constant of integration).

80. (a) $\int 2\sec^2 x\tan x\,dx = \int 2u\,du = u^2 + C = \tan^2 x + C$

(b) $\int 2\sec^2 x\tan x\,dx = \int 2u\,du = u^2 + C = \sec^2 x + C$

(c) Since $\sec^2 x - \tan^2 x = 1$, the two answers differ by a constant (accounted for in the constant of integration).

81. (a) $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos u\,du}{\sqrt{1-\sin^2 u}} = \int \frac{\cos u\,du}{\sqrt{\cos^2 u}} = \int 1\,du.$

(Note $\cos u > 0$, so $\sqrt{\cos^2 u} = |\cos u| = \cos u$.)

(b) $\int \frac{dx}{\sqrt{1-x^2}} = \int 1\,du = u + C = \sin^{-1} x + C$

82. (a) $\int \frac{dx}{1+x^2} = \int \frac{\sec^2 u\,du}{1+\tan^2 u} = \int \frac{\sec^2 u\,du}{\sec^2 u} = \int 1\,du$

(b) $\int \frac{dx}{1+x^2} = \int 1\,du = u + C = \tan^{-1} x + C$

83. (a) $\int_0^{1/2} \frac{\sqrt{x}\,dx}{\sqrt{1-x}} = \int_{\sin^{-1}\sqrt{0}}^{\sin^{-1}\sqrt{1/2}} \frac{\sin y\cdot 2\sin y\cos y\,dy}{\sqrt{1-\sin^2 y}}$

$= \int_0^{\pi/4} \frac{2\sin^2 y\cos y\,dy}{\cos y} = \int_0^{\pi/4} 2\sin^2 y\,dy$

(b) $\int_0^{1/2} \frac{\sqrt{x}\,dx}{\sqrt{1-x}} = \int_0^{\pi/4} 2\sin^2 y\,dy = \int_0^{\pi/4} (1-\cos 2y)\,dy$

$= [y - (1/2)\sin 2y]\Big|_0^{\pi/4} = (\pi - 2)/4$

84. (a) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}} = \int_{\tan^{-1}(0)}^{\tan^{-1}(\sqrt{3})} \frac{\sec^2 u\,du}{\sqrt{1+\tan^2 u}} = \int_0^{\pi/3} \frac{\sec^2 u\,du}{\sec u} = \int_0^{\pi/3} \sec u\,du$

(b) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{1+x^2}} = \int_0^{\pi/3} \sec u\,du = \Big[\ln|\sec u + \tan u|\Big]_0^{\pi/3}$

$= \ln(2+\sqrt{3}) - \ln(1+0) = \ln(2+\sqrt{3})$

6.3 Antidifferentiation by Parts

What you'll learn about

- Product Rule in Integral Form
- Solving for the Unknown Integral
- Tabular Integration
- Inverse Trigonometric and Logarithmic Functions

. . . and why

The Product Rule relates to derivatives as the technique of parts relates to antiderivatives.

Product Rule in Integral Form

When u and v are differentiable functions of x, the Product Rule for differentiation tells us that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides with respect to x and rearranging leads to the integral equation

$$\int \left(u\frac{dv}{dx}\right) dx = \int \left(\frac{d}{dx}(uv)\right) dx - \int \left(v\frac{du}{dx}\right) dx$$

$$= uv - \int \left(v\frac{du}{dx}\right) dx.$$

When this equation is written in the simpler differential notation we obtain the following formula.

Integration by Parts Formula

$$\int u\, dv = uv - \int v\, du$$

This formula expresses one integral, $\int u\, dv$, in terms of a second integral, $\int v\, du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. This is the reason for the importance of the formula. When faced with an integral that we cannot handle analytically, we can replace it by one with which we might have more success.

L I P E T

If you are wondering what to choose for u, here is what we usually do. Our first choice is a natural logarithm (L), if there is one. If there isn't, we look for an inverse trigonometric function (I). If there isn't one of these either, look for a polynomial (P). Barring that, look for an exponential (E) or a trigonometric function (T). That's the preference order: **L I P E T.**

In general, we want u to be something that simplifies when differentiated, and dv to be something that remains manageable when integrated.

EXAMPLE 1 Using Integration by Parts

Evaluate $\int x \cos x\, dx$.

SOLUTION

We use the formula $\int u\, dv = uv - \int v\, du$ with

$$u = x, \qquad dv = \cos x\, dx.$$

To complete the formula, we take the differential of u and find the simplest antiderivative of $\cos x$.

$$du = dx \qquad v = \sin x$$

Then,

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

Now try Exercise 1.

Let's examine the choices available for u and v in Example 1.

EXPLORATION 1 **Choosing the Right u and dv**

Not every choice of u and dv leads to success in antidifferentiation by parts. There is always a trade-off when we replace $\int u\,dv$ with $\int v\,du$, and we gain nothing if $\int v\,du$ is no easier to find than the integral we started with. Let us look at the other choices we might have made in Example 1 to find $\int x \cos x\,dx$.

1. Apply the parts formula to $\int x \cos x\,dx$, letting $u = 1$ and $dv = x \cos x\,dx$. Analyze the result to explain why the choice of $u = 1$ is never a good one.
2. Apply the parts formula to $\int x \cos x\,dx$, letting $u = x \cos x$ and $dv = dx$. Analyze the result to explain why this is not a good choice for this integral.
3. Apply the parts formula to $\int x \cos x\,dx$, letting $u = \cos x$ and $dv = x\,dx$. Analyze the result to explain why this is not a good choice for this integral.
4. What makes x a good choice for u and $\cos x\,dx$ a good choice for dv?

The goal of integration by parts is to go from an integral $\int u\,dv$ that we don't see how to evaluate to an integral $\int v\,du$ that we can evaluate. Keep in mind that integration by parts does not always work.

Sometimes we have to use integration by parts more than once to evaluate an integral.

EXAMPLE 2 Repeated Use of Integration by Parts

Evaluate $\int x^2 e^x\,dx$.

SOLUTION

With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int x e^x\,dx \\ &= x^2 e^x - 2x e^x + 2e^x + C.\end{aligned}$$

The technique of Example 2 works for any integral $\int x^n e^x\,dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy. We will say more on this later in this section when we discuss *tabular integration.*

Now try Exercise 5.

EXAMPLE 3 Solving an Initial Value Problem

Solve the differential equation $dy/dx = x \ln(x)$ subject to the initial condition $y = -1$ when $x = 1$. Confirm the solution graphically by showing that it conforms to the slope field.

continued

SOLUTION

We find the antiderivative of $x \ln(x)$ by using parts. It is usually a better idea to differentiate $\ln(x)$ than to antidifferentiate it (do you see why?), so we let $u = \ln(x)$ and $dv = x\,dx$.

$$\begin{aligned} y &= \int x \ln(x)\,dx \\ &= \left(\frac{x^2}{2}\right)\ln(x) - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx \\ &= \left(\frac{x^2}{2}\right)\ln(x) - \int\left(\frac{x}{2}\right)dx \\ &= \left(\frac{x^2}{2}\right)\ln(x) - \frac{x^2}{4} + C \end{aligned}$$

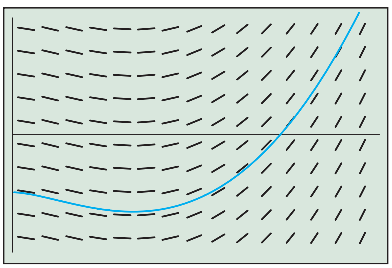
Using the initial condition,

$$\begin{aligned} -1 &= \left(\frac{1}{2}\right)\ln(1) - \frac{1}{4} + C \\ -\frac{3}{4} &= 0 + C \\ C &= -\frac{3}{4}. \end{aligned}$$

Thus

$$y = \left(\frac{x^2}{2}\right)\ln(x) - \frac{x^2}{4} - \frac{3}{4}.$$

Figure 6.9 shows a graph of this function superimposed on a slope field for $dy/dx = x\ln(x)$, to which it conforms nicely. **Now try Exercise 11.**



[0, 3] by [–1.5, 1.5]

Figure 6.9 The solution to the initial value problem in Example 3 conforms nicely to a slope field of the differential equation. (Example 3)

Solving for the Unknown Integral

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Solving for the Unknown Integral

Evaluate $\int e^x \cos x\,dx$.

SOLUTION

Let $u = e^x$, $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The second integral is like the first, except it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x\,dx, \quad v = -\cos x, \quad du = e^x\,dx.$$

Then

$$\begin{aligned} \int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int(-\cos x)(e^x\,dx)\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx. \end{aligned}$$

continued

The unknown integral now appears on both sides of the equation. Adding the integral to both sides gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Now try Exercise 17.

When making repeated use of integration by parts in circumstances like Example 4, once a choice for u and dv is made, it is usually not a good idea to switch choices in the second stage of the problem. Doing so will result in undoing the work. For example, if we had switched to the substitution $u = \sin x$, $dv = e^x dx$ in the second integration, we would have obtained

$$\int e^x \cos x\,dx = e^x \sin x - \left(e^x \sin x - \int e^x \cos x\,dx\right)$$
$$= \int e^x \cos x\,dx,$$

undoing the first integration by parts.

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is **tabular integration,** as shown in Examples 5 and 6.

EXAMPLE 5 Using Tabular Integration

Evaluate $\int x^2 e^x\,dx$.

SOLUTION

With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 2. ***Now try Exercise 21.***

EXAMPLE 6 Using Tabular Integration

Evaluate $\int x^3 \sin x\, dx$.

SOLUTION

With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Now try Exercise 23.

Inverse Trigonometric and Logarithmic Functions

The method of parts is only useful when the integrand can be written as a product of two functions (u and dv). In fact, *any* integrand $f(x)\, dx$ satisfies that requirement, since we can let $u = f(x)$ and $dv = dx$. There are not many antiderivatives of the form $\int f(x)\, dx$ that you would want to find by parts, but there are some, most notably the antiderivatives of logarithmic and inverse trigonometric functions.

EXAMPLE 7 Antidifferentiating ln *x*

Find $\int \ln x\, dx$.

SOLUTION

If we want to use parts, we have little choice but to let $u = \ln x$ and $dv = dx$.

$$\int \ln x\, dx = (\ln x)(x) - \int (x)\left(\frac{1}{x}\right) dx \qquad \int u\, dv = uv - \int v\, du$$

$$= x \ln x - \int 1\, dx$$

$$= x \ln x - x + C$$

EXAMPLE 8 Antidifferentiating $\sin^{-1} x$

Find the solution to the differential equation $dy/dx = \sin^{-1} x$ if the graph of the solution passes through the point (0, 0).

SOLUTION

We find $\int \sin^{-1} x\, dx$, letting $u = \sin^{-1} x$, $dv = dx$.

continued

$$\int \sin^{-1} x\,dx = (\sin^{-1} x)(x) - \int (x)\left(\frac{1}{\sqrt{1-x^2}}\right) dx \qquad \int u\,dv = uv - \int v\,du$$

$$= x \sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

$$= x \sin^{-1} x + \frac{1}{2}\int \frac{-2x\,dx}{\sqrt{1-x^2}} \qquad \text{Set up } u\text{-substitution.}$$

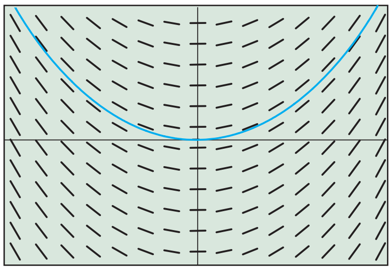
$$= x \sin^{-1} x + \frac{1}{2}\int u^{-1/2} du \qquad \text{Let } u = 1 - x^2,\ du = -2x\,dx.$$

$$= x \sin^{-1} x + u^{1/2} + C$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C \qquad \text{Re-substitute.}$$

Applying the initial condition $y = 0$ when $x = 0$, we conclude that the particular solution is $y = x \sin^{-1} x + \sqrt{1-x^2} - 1$.

A graph of $y = x \sin^{-1} x + \sqrt{1-x^2} - 1$ conforms nicely to the slope field for $dy/dx = \sin^{-1} x$, as shown in Figure 6.10.



[–1, 1] by [–0.5, 0.5]

Figure 6.10 The solution to the initial value problem in Example 8 conforms nicely to the slope field of the differential equation. (Example 8)

9. $y = \frac{1}{2}x^2 - \cos x + 3$

Quick Review 6.3 *(For help, go to Sections 3.8 and 3.9.)*

In Exercises 1–4, find dy/dx.

1. $y = x^3 \sin 2x$ $\quad 2x^3 \cos 2x + 3x^2 \sin 2x$

2. $y = e^{2x} \ln(3x+1)$ $\quad \frac{3e^{2x}}{3x+1} + 2e^{2x} \ln(3x+1)$

3. $y = \tan^{-1} 2x$ $\quad \frac{2}{1+4x^2}$

4. $y = \sin^{-1}(x+3)$ $\quad \frac{1}{\sqrt{1-(x+3)^2}}$

In Exercises 5 and 6, solve for x in terms of y.

5. $y = \tan^{-1} 3x$ $\quad x = \frac{1}{3}\tan y$

6. $y = \cos^{-1}(x+1)$ $\quad x = \cos y - 1$

7. Find the area under the arch of the curve $y = \sin \pi x$ from $x = 0$ to $x = 1$. $\quad \frac{2}{\pi}$

8. Solve the differential equation $dy/dx = e^{2x}$. $\quad y = \frac{1}{2}e^{2x} + C$

9. Solve the initial value problem $dy/dx = x + \sin x$, $\quad y(0) = 2$.

10. Use differentiation to confirm the integration formula

$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x).$$

$$\frac{d}{dx}\left[\frac{1}{2}e^x(\sin x - \cos x)\right] = e^x \sin x$$

Section 6.3 Exercises

In Exercises 1–10, find the indefinite integral.

1. $\int x \sin x\,dx$ $\quad -x\cos x + \sin x + C$

2. $\int x\,e^x\,dx$ $\quad xe^x - e^x + C$

3. $\int 3t\,e^{2t}\,dt$ $\quad \frac{3}{2}te^{2t} - \frac{3}{4}e^{2t} + C$

4. $\int 2t\cos(3t)\,dt$ $\quad \frac{2}{3}t\sin(3t) + \frac{2}{9}\cos(3t) + C$

5. $\int x^2 \cos x\,dx$ $\quad x^2 \sin x + 2x\cos x - 2\sin x + C$

6. $\int x^2 e^{-x}\,dx$ $\quad -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

7. $\int 3x^2 e^{2x}\,dx$

8. $\int x^2 \cos\left(\frac{x}{2}\right) dx$

9. $\int y \ln y\,dy$ $\quad \frac{y^2}{2}\ln y - \frac{y^2}{4} + C$

10. $\int t^2 \ln t\,dt$ $\quad \frac{t^3}{3}\ln t - \frac{t^3}{9} + C$

7. $\frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$ **8.** $2x^2 \sin\left(\frac{x}{2}\right) + 8x\cos\left(\frac{x}{2}\right) - 16\sin\left(\frac{x}{2}\right) + C$

In Exercises 11–16, solve the initial value problem. Confirm your answer by checking that it conforms to the slope field of the differential equation.

11. $\frac{dy}{dx} = (x+2)\sin x$ and $y = 2$ when $x = 0$

12. $\frac{dy}{dx} = 2xe^{-x}$ and $y = 3$ when $x = 0$

13. $\frac{du}{dx} = x\sec^2 x$ and $u = 1$ when $x = 0$

14. $\frac{dz}{dx} = x^3 \ln x$ and $z = 5$ when $x = 1$

15. $\frac{dy}{dx} = x\sqrt{x-1}$ and $y = 2$ when $x = 1$

16. $\frac{dy}{dx} = 2x\sqrt{x+2}$ and $y = 0$ when $x = -1$

19. $\frac{e^x}{5}(2\sin 2x + \cos 2x) + C$ **20.** $-\frac{e^{-x}}{5}(2\cos 2x + \sin 2x) + C$

In Exercises 17–20, use parts and solve for the unknown integral.

17. $\int e^x \sin x\, dx$ $\frac{e^x}{2}(\sin x - \cos x) + C$

18. $\int e^{-x} \cos x\, dx$ $\frac{e^{-x}}{2}(\sin x - \cos x) + C$

19. $\int e^x \cos 2x\, dx$

20. $\int e^{-x} \sin 2x\, dx$

In Exercises 21–24, use tabular integration to find the antiderivative.

21. $\int x^4 e^{-x}\, dx$ $(-x^4 - 4x^3 - 12x^2 - 24x - 24)e^{-x} + C$

22. $\int (x^2 - 5x)e^x\, dx$ $(x^2 - 7x + 7)e^x + C$

23. $\int x^3 e^{-2x}\, dx$ See page 348.

24. $\int x^3 \cos 2x\, dx$ See page 348.

In Exercises 25–28, evaluate the integral analytically. Support your answer using NINT.

25. $\int_0^{\pi/2} x^2 \sin 2x\, dx$ $\frac{\pi^2}{8} - \frac{1}{2} \approx 0.734$

26. $\int_0^{\pi/2} x^3 \cos 2x\, dx$ $\frac{3}{4} - \frac{3\pi^2}{16} \approx -1.101$

27. $\int_{-2}^{3} e^{2x} \cos 3x\, dx$ See page 348.

28. $\int_{-3}^{2} e^{-2x} \sin 2x\, dx$ See page 348.

In Exercises 29–32, solve the differential equation.

29. $\frac{dy}{dx} = x^2 e^{4x}$ See page 348.

30. $\frac{dy}{dx} = x^2 \ln x$ See page 348.

31. $\frac{dy}{d\theta} = \theta \sec^{-1}\theta, \quad \theta > 1$ See page 348.

32. $\frac{dy}{d\theta} = \theta \sec\theta \tan\theta$ See page 348.

33. *Finding Area* Find the area of the region enclosed by the x-axis and the curve $y = x \sin x$ for

(a) $0 \le x \le \pi$, π **(b)** $\pi \le x \le 2\pi$, 3π **(c)** $0 \le x \le 2\pi$. 4π

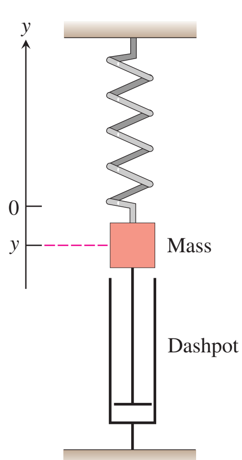
34. *Finding Area* Find the area of the region enclosed by the y-axis and the curves $y = x^2$ and $y = (x^2 + x + 1)e^{-x}$. ≈ 0.726

35. *Average Value* A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t}\cos t, \quad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

$\frac{1 - e^{-2\pi}}{2\pi} \approx 0.159$



Standardized Test Questions

You should solve the following problems without using a graphing calculator.

36. True or False If $f'(x) = g(x)$, then $\int x\, g(x)\, dx = x\, f(x) - \int f(x)\, dx$. Justify your answer. See page 348.

37. True or False If $f'(x) = g(x)$, then $\int x^2\, g(x)\, dx = x^2 f(x) - 2\int x\, f(x)\, dx$. Justify your answer. See page 348.

38. Multiple Choice If $\int x^2 \cos x\, dx = h(x) - \int 2x \sin x\, dx$, then $h(x) =$ B

(A) $2\sin x + 2x\cos x + C$

(B) $x^2 \sin x + C$

(C) $2x\cos x - x^2 \sin x + C$

(D) $4\cos x - 2x\sin x + C$

(E) $(2 - x^2)\cos x - 4\sin x + C$

39. Multiple Choice $\int x \sin(5x)\, dx =$ B

(A) $-x\cos(5x) + \sin(5x) + C$

(B) $-\frac{x}{5}\cos(5x) + \frac{1}{25}\sin(5x) + C$

(C) $-\frac{x}{5}\cos(5x) + \frac{1}{5}\sin(5x) + C$

(D) $\frac{x}{5}\cos(5x) + \frac{1}{25}\sin(5x) + C$

(E) $5x\cos(5x) - \sin(5x) + C$

40. Multiple Choice $\int x \csc^2 x\, dx =$ C

(A) $\frac{x^2 \csc^3 x}{6} + C$

(B) $x\cot x - \ln|\sin x| + C$

(C) $-x\cot x + \ln|\sin x| + C$

(D) $-x\cot x - \ln|\sin x| + C$

(E) $-x\sec^2 x - \tan x + C$

41. Multiple Choice The graph of $y = f(x)$ conforms to the slope field for the differential equation $dy/dx = 4x \ln x$, as shown in the graph below. Which of the following could be $f(x)$? C

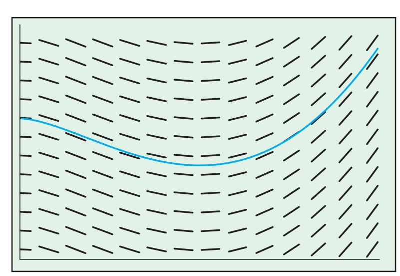
(A) $2x^2(\ln x)^2 + 3$

(B) $x^3 \ln x + 3$

(C) $2x^2 \ln x - x^2 + 3$

(D) $(2x^2 + 3)\ln x - 1$

(E) $2x(\ln x)^2 - \frac{4}{3}(\ln x)^3 + 3$



[0, 2] by [0, 5]

Explorations

42. Consider the integral $\int x^n e^x\,dx$. Use integration by parts to evaluate the integral if

(a) $n = 1$. $(x - 1)e^x + C$

(b) $n = 2$. $(x^2 - 2x + 2)e^x + C$

(c) $n = 3$. $(x^3 - 3x^2 + 6x - 6)e^x + C$

See below.

(d) Conjecture the value of the integral for any positive integer n.

(e) Writing to Learn Give a convincing argument that your conjecture in part (d) is true. Use mathematical induction or argue based on tabular integration.

In Exercises 43–46, evaluate the integral by using a substitution prior to integration by parts.

43. $\int \sin \sqrt{x}\,dx$ $-2(\sqrt{x}\cos\sqrt{x} - \sin\sqrt{x}) + C$

44. $\int e^{\sqrt{3x+9}}\,dx$

45. $\int x^7 e^{x^2}\,dx$

46. $\int \sin(\ln r)\,dr$ $\frac{r}{2}[\sin(\ln r) - \cos(\ln r)] + C$

In Exercises 47–50, use integration by parts to establish the *reduction formula.*

47. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$ $u = x^n,\ dv = \cos x\,dx$

48. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$ $u = x^n,\ dv = \sin x\,dx$

49. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$ $u = x^n,\ dv = e^{ax}\,dx$

50. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$ $u = (\ln x)^n,\ dv = dx$

Extending the Ideas

51. *Integrating Inverse Functions* Assume that the function f has an inverse.

(a) Show that $\int f^{-1}(x)\,dx = \int y f'(y)\,dy$. (*Hint:* Use the substitution $y = f^{-1}(x)$.)

(b) Use integration by parts on the second integral in part (a) to show that

$$\int f^{-1}(x)\,dx = \int y f'(y)\,dy = x f^{-1}(x) - \int f(y)\,dy.$$

52. *Integrating Inverse Functions* Assume that the function f has an inverse. Use integration by parts directly to show that

$$\int f^{-1}(x)\,dx = x f^{-1}(x) - \int x\left(\frac{d}{dx} f^{-1}(x)\right)dx.$$

$u = f^{-1}(x),\ dv = dx$

In Exercises 53–56, evaluate the integral using

(a) the technique of Exercise 51.

(b) the technique of Exercise 52.

(c) Show that the expressions (with $C = 0$) obtained in parts (a) and (b) are the same.

53. $\int \sin^{-1} x\,dx$

54. $\int \tan^{-1} x\,dx$

55. $\int \cos^{-1} x\,dx$

56. $\int \log_2 x\,dx$

Answers:

23. $\left(-\frac{x^3}{2} - \frac{3x^2}{4} - \frac{3x}{4} - \frac{3}{8}\right)e^{-2x} + C$

24. $\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x + C$

27. $\frac{1}{13}[e^6(2\cos 9 + 3\sin 9) - e^{-4}(2\cos 6 - 3\sin 6)] \approx -18.186$

28. $-\frac{e^{-4}}{4}(\cos 4 + \sin 4) + \frac{e^6}{4}(\cos 6 - \sin 6) \approx 125.03$

29. $y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right)e^{4x} + C$ **30.** $y = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

31. $y = \frac{\theta^2}{2}\sec^{-1}\theta - \frac{1}{2}\sqrt{\theta^2 - 1} + C$

32. $y = \theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$

36. True. Use parts, letting $u = x$, $dv = g(x)dx$, and $v = f(x)$.

37. True. Use parts, letting $u = x^2$, $dv = g(x)dx$, and $v = f(x)$.

42. (d) $\left[x^n - \frac{d(x^n)}{dx} + \frac{d^2(x^n)}{dx^2} - \cdots + (-1)^n\frac{d^n(x^n)}{dx^n}\right]e^x + C$ or $[x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^{n-1}(n!)x + (-1)^n(n!)]e^x + C$

44. $\frac{2(\sqrt{3x+9} - 1)e^{\sqrt{3x+9}}}{3} + C$ **45.** $\frac{(x^6 - 3x^4 + 6x^2 - 6)e^{x^2}}{2} + C$

51. (a) Let $y = f^{-1}(x)$. Then $x = f(y)$, so $dx = f'(y)\,dy$. Substitute directly.
(b) $u = y,\ dv = f'(y)\,dy$

53. (a) $\int \sin^{-1} x\,dx = x\sin^{-1} x + \cos(\sin^{-1} x) + C$

(b) $\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1 - x^2} + C$

(c) $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

54. (a) $\int \tan^{-1} x\,dx = x\tan^{-1} x + \ln|\cos(\tan^{-1} x)| + C$

(b) $\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1 + x^2) + C$

(c) $\ln|\cos(\tan^{-1} x)| = \ln\left(\frac{1}{\sqrt{1 + x^2}}\right) = -\frac{1}{2}\ln(1 + x^2)$

55. (a) $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$

(b) $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1 - x^2} + C$

(c) $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$

56. (a) $\int \log_2 x\,dx = x\log_2 x - \left(\frac{1}{\ln 2}\right)2^{\log_2 x} + C$

(b) $\int \log_2 x\,dx = x\log_2 x - \left(\frac{1}{\ln 2}\right)x + C$

(c) $2^{\log_2 x} = x$

Quick Quiz for AP* Preparation: Sections 6.1–6.3

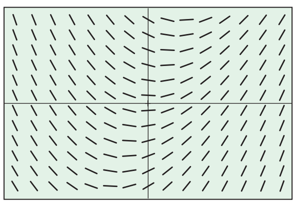
You should solve the following problems without using a graphing calculator.

1. Multiple Choice Which of the following differential equations would produce the slope field shown below? E

(A) $\frac{dy}{dx} = y - 3x$ **(B)** $\frac{dy}{dx} = y - \frac{x}{3}$

(C) $\frac{dy}{dx} = y + \frac{x}{3}$ **(D)** $\frac{dy}{dx} = y + \frac{x}{3}$

(E) $\frac{dy}{dx} = x - \frac{y}{3}$



2. Multiple Choice If the substitution $\sqrt{x} = \sin y$ is made in the integrand of $\int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$, the resulting integral is C

(A) $\int_0^{1/2} \sin^2 y\,dy$ **(B)** $2\int_0^{1/2} \frac{\sin^2 y}{\cos y}\,dy$

(C) $2\int_0^{\pi/4} \sin^2 y\,dy$ **(D)** $\int_0^{\pi/4} \sin^2 y\,dy$

(E) $2\int_0^{\pi/6} \sin^2 y\,dy$

3. Multiple Choice $\int x\,e^{2x}\,dx =$ A

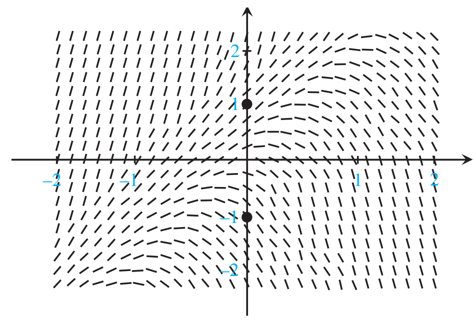
(A) $\frac{x\,e^{2x}}{2} - \frac{e^{2x}}{4} + C$ **(B)** $\frac{x\,e^{2x}}{2} - \frac{e^{2x}}{2} + C$

(C) $\frac{x\,e^{2x}}{2} + \frac{e^{2x}}{4} + C$ **(D)** $\frac{x\,e^{2x}}{2} + \frac{e^{2x}}{2} + C$

(E) $\frac{x^2\,e^{2x}}{4} + C$

4. Free Response Consider the differential equation $dy/dx = 2y - 4x$.

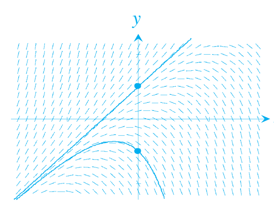
(a) The slope field for the differential equation is shown below. Sketch the solution curve that passes through the point $(0, 1)$ and sketch the solution curve that goes through the point $(0, -1)$.



(b) There is a value of b for which $y = 2x + b$ is a solution to the differential equation. Find this value of b. Justify your answer.

(c) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. It appears from the slope field that g has a local maximum at the point $(0, 0)$. Using the differential equation, prove analytically that this is so.

4. (a)



(b) Let $\frac{dy}{dx} = 2$ and $y = 2x + b$ in the differential equation:

$2 = 2(2x + b) - 4x$

$2 = 2b$

$b = 1$

(c) First, note that $\frac{dy}{dx} = 2(0) - 4(0) = 0$ at the point $(0, 0)$.

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}(2y - 4x) = 2\frac{dy}{dx} - 4$, which is -4 at the point $(0, 0)$.

By the Second Derivative test, g has a local maximum at $(0, 0)$.

6.4 Exponential Growth and Decay

What you'll learn about

- Separable Differential Equations
- Law of Exponential Change
- Continuously Compounded Interest
- Radioactivity
- Modeling Growth with Other Bases
- Newton's Law of Cooling

. . . and why

Understanding the differential equation $dy/dx = ky$ gives us new insight into exponential growth and decay.

Separable Differential Equations

Before we revisit the topic of exponential growth (last seen as a precalculus topic in Chapter P), we need to introduce the concept of separable differential equations.

DEFINITION Separable Differential Equation

A differential equation of the form $dy/dx = f(y)g(x)$ is called **separable.** We **separate the variables** by writing it in the form

$$\frac{1}{f(y)}\,dy = g(x)\,dx.$$

The solution is found by antidifferentiating each side with respect to its thusly isolated variable.

EXAMPLE 1 Solving by Separation of Variables

Solve for y if $dy/dx = (xy)^2$ and $y = 1$ when $x = 1$.

SOLUTION

The equation is separable because it can be written in the form $dy/dx = y^2x^2$, where $f(y) = y^2$ and $g(x) = x^2$. We separate the variables and antidifferentiate as follows.

$$y^{-2}\,dy = x^2\,dx \qquad \text{Separate the variables.}$$

$$\int y^{-2}\,dy = \int x^2\,dx \qquad \text{Prepare to antidifferentiate.}$$

$$-y^{-1} = \frac{x^3}{3} + C \qquad \text{Note that only one constant is needed.}$$

We then apply the initial condition to find C.

$$-1 = \frac{1}{3} + C \Rightarrow C = -\frac{4}{3}$$

$$-y^{-1} = \frac{x^3}{3} - \frac{4}{3}$$

$$y^{-1} = \frac{4 - x^3}{3}$$

$$y = \frac{3}{4 - x^3}$$

This solution is valid for the continuous section of the function that goes through the point (1, 1), that is, on the domain $(-\infty, \sqrt[3]{4})$.

It is apparent that $y = 1$ when $x = 1$, but it is worth checking that $dy/dx = (xy)^2$.

$$y = \frac{3}{4 - x^3}$$

$$\frac{dy}{dx} = -3(4 - x^3)^{-2}(-3x^2)$$

$$\frac{dy}{dx} = \frac{9x^2}{(4 - x^3)^2} = x^2\left(\frac{3}{4 - x^3}\right)^2 = x^2y^2 = (xy)^2$$

Now try Exercise 1.

Law of Exponential Change

You have probably solved enough exponential growth problems by now to recognize that they involve growth in which the rate of change is proportional to the amount present. The more bacteria in the dish, the faster they multiply. The more radioactive material present, the faster it decays. The greater your bank account (assuming it earns compounded interest), the faster it grows.

The differential equation that describes this growth is $dy/dt = ky$, where k is the *growth constant* (if positive) or the *decay constant* (if negative). We can solve this equation by separating the variables.

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}\,dy = k\,dt \qquad \text{Separate the variables}$$

$$\ln|y| = kt + C \qquad \text{Antidifferentiate both sides}$$

$$|y| = e^{kt+C} \qquad \text{Exponentiate both sides}$$

$$|y| = e^{C}\,e^{kt} \qquad \text{Property of exponents}$$

$$y = \pm e^{C}\,e^{kt} \qquad \text{Definition of absolute value}$$

$$y = Ae^{kt} \qquad \text{Let } A = \pm e^{C}.$$

What if $A = 0$?

If $A = 0$, then the solution to $dy/dt = ky$ is the constant function $y = 0$. This function is technically of the form $y = Ae^{kt}$, but we do not consider it to be an exponential function. The initial condition in this case leads to a "trivial" solution.

This solution shows that the *only* growth function that results in a growth rate proportional to the amount present is, in fact, exponential. Note that the constant A is the amount present when $t = 0$, so it is usually denoted y_0.

The Law of Exponential Change

If y changes at a rate proportional to the amount present (that is, if $dy/dt = ky$), and if $y = y_0$ when $t = 0$, then

$$y = y_0\,e^{kt}.$$

The constant k is the **growth constant** if $k > 0$ or the **decay constant** if $k < 0$.

Now try Exercise 11.

Continuously Compounded Interest

Suppose that A_0 dollars are invested at a fixed annual interest rate r (expressed as a decimal). If interest is added to the account k times a year, the amount of money present after t years is

$$A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}.$$

The interest might be added ("compounded," bankers say) monthly ($k = 12$), weekly ($k = 52$), daily ($k = 365$), or even more frequently, by the hour or by the minute.

If, instead of being added at discrete intervals, the interest is added continuously at a rate proportional to the amount in the account, we can model the growth of the account with the initial value problem.

$$\text{Differential equation:} \quad \frac{dA}{dt} = rA$$

$$\text{Initial condition:} \quad A(0) = A_0$$

It can be shown that

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} = A_0e^{rt}.$$

We will see how this limit is evaluated in Section 8.2, Exercise 57.

The amount of money in the account after t years is then

$$A(t) = A_0e^{rt}.$$

Interest paid according to this formula is said to be **compounded continuously.** The number r is the **continuous interest rate.**

EXAMPLE 2 Compounding Interest Continuously

Suppose you deposit \$800 in an account that pays 6.3% annual interest. How much will you have 8 years later if the interest is **(a)** compounded continuously? **(b)** compounded quarterly?

SOLUTION

Here $A_0 = 800$ and $r = 0.063$. The amount in the account to the nearest cent after 8 years is

(a) $A(8) = 800e^{(0.063)(8)} = 1324.26.$

(b) $A(8) = 800\left(1+\dfrac{0.063}{4}\right)^{(4)(8)} = 1319.07.$

You might have expected to generate more than an additional \$5.19 with interest compounded continuously. *Now try Exercise 19.*

For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice for the painters, who licked their brush-tips), t is measured in years and $k = 4.3 \times 10^{-4}$. For radon-222 gas, t is measured in days and $k = 0.18$. The decay of radium in the earth's crust is the source of the radon we sometimes find in our basements.

Radioactivity

When an atom emits some of its mass as radiation, the remainder of the atom reforms to make an atom of some new element. This process of radiation and change is **radioactive decay,** and an element whose atoms go spontaneously through this process is **radioactive.** Radioactive carbon-14 decays into nitrogen. Radium, through a number of intervening radioactive steps, decays into lead.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit of time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0e^{-kt}, \qquad k > 0.$$

Convention

It is conventional to use $-k$ $(k > 0)$ here instead of k $(k < 0)$ to emphasize that y is decreasing.

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. Example 3 shows the surprising fact that the half-life is a constant that depends only on the radioactive substance and not on the number of radioactive nuclei present in the sample.

EXAMPLE 3 Finding Half-Life

Find the half-life of a radioactive substance with decay equation $y = y_0e^{-kt}$ and show that the half-life depends only on k.

SOLUTION

Model The half-life is the solution to the equation

$$y_0e^{-kt} = \frac{1}{2}y_0.$$

continued

Solve Algebraically

$$e^{-kt} = \frac{1}{2} \quad \text{Divide by } y_0.$$

$$-kt = \ln \frac{1}{2} \quad \text{Take ln of both sides.}$$

$$t = -\frac{1}{k}\ln\frac{1}{2} = \frac{\ln 2}{k} \quad \ln\frac{1}{a} = -\ln a$$

Interpret This value of t is the half-life of the element. It depends only on the value of k. Note that the number y_0 does not appear. ***Now try Exercise 21.***

DEFINITION Half-life

The **half-life** of a radioactive substance with rate constant k $(k > 0)$ is

$$\text{half-life} = \frac{\ln 2}{k}.$$

Modeling Growth with Other Bases

As we have seen, the differential equation $dy/dt = ky$ leads to the exponential solution

$$y = y_0\, e^{kt},$$

where y_0 is the value of y at $t = 0$. We can also write this solution in the form

$$y = y_0\, b^{ht},$$

where b is any positive number not equal to 1, and h is another rate constant, related to k by the equation $k = h \ln b$. This means that exponential growth can be modeled in *any* positive base not equal to 1, enabling us to choose a convenient base to fit a given growth pattern, as the following exploration shows.

EXPLORATION 1 Choosing a Convenient Base

A certain population y is growing at a continuous rate so that the population doubles every 5 years.

1. Let $y = y_0\, 2^{ht}$. Since $y = 2\, y_0$ when $t = 5$, what is h? What is the relationship of h to the doubling period?
2. How long does it take for the population to triple?

A certain population y is growing at a continuous rate so that the population triples every 10 years.

3. Let $y = y_0\, 3^{ht}$. Since $y = 3\, y_0$ when $t = 10$, what is h? What is the relationship of h to the tripling period?
4. How long does it take for the population to double?

A certain isotope of sodium (Na-24) has a half-life of 15 hours. That is, half the atoms of Na-24 disintegrate into another nuclear form in fifteen hours.

5. Let $A = A_0(1/2)^{ht}$. Since $y = (1/2)\, y_0$ when $t = 15$, what is h? What is the relationship of h to the half-life?
6. How long does it take for the amount of radioactive material to decay to 10% of the original amount?

It is important to note that while the exponential growth model $y = y_0\, b^{ht}$ satisfies the differential equation $dy/dt = ky$ for any positive base b, it is only when $b = e$ that the growth constant k appears in the exponent as the coefficient of t. In general, the coefficient of t is the reciprocal of the time period required for the population to grow (or decay) by a factor of b.

Hornets Aplenty

The hornet population in Example 4 can grow exponentially for a while, but the ecosystem cannot sustain such growth all summer. The model will eventually become logistic, as we will see in the next section. Nevertheless, given the right conditions, a bald-faced hornet nest can grow to be bigger than a basketball and house more than 600 workers.

EXAMPLE 4 Choosing a Base

At the beginning of the summer, the population of a hive of bald-faced hornets (which are actually wasps) is growing at a rate proportional to the population. From a population of 10 on May 1, the number of hornets grows to 50 in thirty days. If the growth continues to follow the same model, how many days after May 1 will the population reach 100?

SOLUTION

Since $dy/dt = ky$, the growth is exponential. Noting that the population grows by a factor of 5 in 30 days, we model the growth in base 5: $y = 10 \times 5^{(1/30)t}$. Now we need only solve the equation $100 = 10 \times 5^{(1/30)t}$ for t:

$$100 = 10 \times 5^{(1/30)t}$$

$$10 = 5^{(1/30)t}$$

$$\ln 10 = (1/30)t \ln 5$$

$$t = 30\left(\frac{\ln 10}{\ln 5}\right) = 42.920$$

Approximately 43 days will pass after May 1 before the population reaches 100.

Now try Exercise 23.

Carbon-14 Dating

The decay of radioactive elements can sometimes be used to date events from earth's past. The ages of rocks more than 2 billion years old have been measured by the extent of the radioactive decay of uranium (half-life 4.5 billion years!). In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's surroundings at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 decreases as the carbon-14 decays. It is possible to estimate the ages of fairly old organic remains by comparing the proportion of carbon-14 they contain with the proportion assumed to have been in the organism's environment at the time it lived. Archeologists have dated shells (which contain $CaCO_3$), seeds, and wooden artifacts this way. The estimate of 15,500 years for the age of the cave paintings at Lascaux, France, is based on carbon-14 dating. After generations of controversy, the Shroud of Turin, long believed by many to be the burial cloth of Christ, was shown by carbon-14 dating in 1988 to have been made after A.D. 1200.

EXAMPLE 5 Using Carbon-14 Dating

Scientists who use carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

SOLUTION

We model the exponential decay in base 1/2: $A = A_0(1/2)^{t/5700}$. We seek the value of t for which $0.9A_0 = A_0(1/2)^{t/5700}$, or $(1/2)^{t/5700} = 0.9$.

Solving algebraically with logarithms,

$$(1/2)^{t/5700} = 0.9$$

$$(t/5700)\ln(1/2) = \ln(0.9)$$

$$t = 5700\left(\frac{\ln(0.9)}{\ln(0.5)}\right)$$

$$t \approx 866.$$

Interpreting the answer, we conclude that the sample is about 866 years old.

Now try Exercise 25.

Newton's Law of Cooling

Soup left in a cup cools to the temperature of the surrounding air. A hot silver ingot immersed in water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium.

J. Ernest Wilkins, Jr.
(1923–)

By the age of nineteen, J. Ernest Wilkins had earned a Ph.D. degree in Mathematics from the University of Chicago. He then taught, served on the Manhattan project (the goal of which was to build the first atomic bomb), and worked as a mathematician and physicist for several corporations. In 1970, Dr. Wilkins joined the faculty at Howard University and served as head of the electrical engineering, physics, chemistry, and mathematics departments before retiring. He is currently working as Distinguished Professor of Applied Mathematics and Mathematical Physics at Clark Atlanta University.

This observation is *Newton's Law of Cooling,* although it applies to warming as well, and there is an equation for it.

If T is the temperature of the object at time t, and T_s is the surrounding temperature, then

$$\frac{dT}{dt} = -k(T - T_s). \tag{1}$$

Since $dT = d(T - T_s)$, Equation 1 can be written as

$$\frac{d}{dt}(T - T_s) = -k(T - T_s).$$

Its solution, by the law of exponential change, is

$$T - T_s = (T_0 - T_s)e^{-kt},$$

where T_0 is the temperature at time $t = 0$. This equation also bears the name **Newton's Law of Cooling.**

EXAMPLE 6 Using Newton's Law of Cooling

A hard-boiled egg at 98°C is put in a pan under running 18°C water to cool. After 5 minutes, the egg's temperature is found to be 38°C. How much longer will it take the egg to reach 20°C?

SOLUTION

Model Using Newton's Law of Cooling with $T_s = 18$ and $T_0 = 98$, we have

$$T - 18 = (98 - 18)e^{-kt} \quad \text{or} \quad T = 18 + 80e^{-kt}.$$

To find k we use the information that $T = 38$ when $t = 5$.

$$38 = 18 + 80e^{-5k}$$

$$e^{-5k} = \frac{1}{4}$$

$$-5k = \ln\frac{1}{4} = -\ln 4$$

$$k = \frac{1}{5}\ln 4$$

The egg's temperature at time t is $T = 18 + 80e^{-(0.2 \ln 4)t}$.

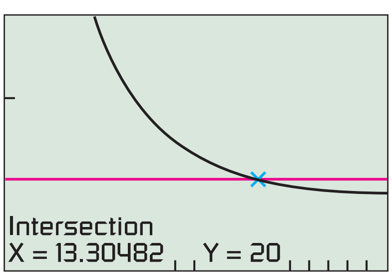
Solve Graphically We can now use a grapher to find the time when the egg's temperature is 20°C. Figure 6.11 shows that the graphs of

$$y = 20 \quad \text{and} \quad y = T = 18 + 80e^{-(0.2 \ln 4)t}$$

intersect at about $t = 13.3$.

Interpret The egg's temperature will reach 20°C in about 13.3 min after it is put in the pan under running water to cool. Because it took 5 min to reach 38°C, it will take slightly more than 8 additional minutes to reach 20°C.

Now try Exercise 31.



[0, 20] by [10, 40]

Figure 6.11 The egg will reach 20°C about 13.3 min after being placed in the pan to cool. (Example 6)

The next example shows how to use exponential regression to fit a function to real data. A CBL™ temperature probe was used to collect the data.

Table 6.1 Experimental Data

Time (sec)	T (°C)	$T - T_s$ (°C)
2	64.8	60.3
5	49.0	44.5
10	31.4	26.9
15	22.0	17.5
20	16.5	12.0
25	14.2	9.7
30	12.0	7.5

EXAMPLE 7 Using Newton's Law of Cooling

A temperature probe (thermometer) is removed from a cup of coffee and placed in water that has a temperature of $T_s = 4.5$°C. Temperature readings T, as recorded in Table 6.1, are taken after 2 sec, 5 sec, and every 5 sec thereafter. Estimate

(a) the coffee's temperature at the time the temperature probe was removed.

(b) the time when the temperature probe reading will be 8°C.

SOLUTION

Model According to Newton's Law of Cooling, $T - T_s = (T_0 - T_s)e^{-kt}$, where $T_s = 4.5$ and T_0 is the temperature of the coffee (probe reading) at $t = 0$.

We use exponential regression to find that

$$T - 4.5 = 61.66(0.9277^t)$$

is a model for the $(t, T - T_s) = (t, T - 4.5)$ data.

Thus,

$$T = 4.5 + 61.66(0.9277^t)$$

is a model for the (t, T) data.

Figure 6.12a shows the graph of the model superimposed on a scatter plot of the (t, T) data.

(a) At time $t = 0$, when the probe was removed, the temperature was

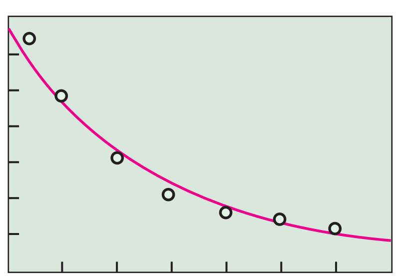
$$T = 4.5 + 61.66(0.9277^0) \approx 66.16°\text{C}.$$

(b) Solve Graphically Figure 6.12b shows that the graphs of

$$y = 8 \quad \text{and} \quad y = T = 4.5 + 61.66(0.9277^t)$$

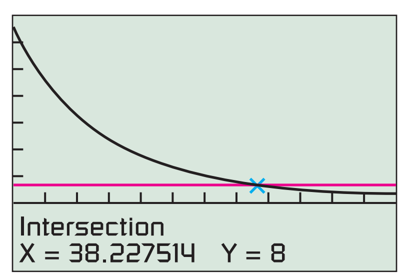
intersect at about $t = 38$.

Interpret The temperature of the coffee was about 66.2°C when the temperature probe was removed. The temperature probe will reach 8°C about 38 sec after it is removed from the coffee and placed in the water.



[0, 35] by [0, 70]

(a)



[0, 60] by [–20, 70]

(b)

Figure 6.12 (Example 7)

Now try Exercise 33.

Quick Review 6.4

You should solve the following problems without using a graphing calculator.

In Exercises 1 and 2, rewrite the equation in exponential form or logarithmic form.

1. $\ln a = b$ $a = e^b$

2. $e^c = d$ $c = \ln(d)$

In Exercises 3–8, solve the equation.

3. $\ln(x + 3) = 2$ $x = e^2 - 3$

4. $100e^{2x} = 600$ $x = \frac{1}{2}\ln 6$

5. $0.85^x = 2.5$ $x = \frac{\ln 2.5}{\ln 0.85} \approx -5.638$

6. $2^{k+1} = 3^k$ $k = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.710$

7. $1.1^t = 10$ $t = \frac{\ln 10}{\ln 1.1} \approx 24.159$

8. $e^{-2t} = \frac{1}{4}$ $t = \frac{1}{2}\ln 4 = \ln 2$

In Exercises 9 and 10, solve for y.

9. $\ln(y + 1) = 2x - 3$ $y = -1 + e^{2x+3}$

10. $\ln|y + 2| = 3t - 1$ $y = -2 \pm e^{3t-1}$

19. (a) 14.94 yr **(b)** 14.62 yr **(c)** 14.68 yr **(d)** 14.59 yr

Section 6.4 Exercises

In Exercises 1–10, use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

1. $\frac{dy}{dx} = \frac{x}{y}$ and $y = 2$ when $x = 1$
$y = \sqrt{x^2 + 3}$, valid for all real numbers

2. $\frac{dy}{dx} = -\frac{x}{y}$ and $y = 3$ when $x = 4$
$y = \sqrt{25 - x^2}$, valid on the interval $(-5, 5)$

3. $\frac{dy}{dx} = \frac{y}{x}$ and $y = 2$ when $x = 2$
$y = x$, valid on the interval $(0, \infty)$

4. $\frac{dy}{dx} = 2xy$ and $y = 3$ when $x = 0$
$y = 3e^{x^2}$, valid for all real numbers

5. $\frac{dy}{dx} = (y + 5)(x + 2)$ and $y = 1$ when $x = 0$
$y = 6e^{x^2/2+2x} - 5$, valid for all real numbers

6. $\frac{dy}{dx} = \cos^2 y$ and $y = 0$ when $x = 0$
$y = \tan^{-1} x$, valid for all real numbers

7. $\frac{dy}{dx} = (\cos x)e^{y+\sin x}$ and $y = 0$ when $x = 0$
$y = -\ln(2 - e^{\sin x})$, valid for all real numbers

8. $\frac{dy}{dx} = e^{x-y}$ and $y = 2$ when $x = 0$
$y = \ln(e^x + e^2 - 1)$, valid for all real numbers

9. $\frac{dy}{dx} = -2xy^2$ and $y = 0.25$ when $x = 1$
$y = (x^2 + 3)^{-1}$, valid for all real numbers

10. $\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}$ and $y = 1$ when $x = e$
$y = (\ln x)^4$, valid on the interval $(0, \infty)$

In Exercises 11–14, find the solution of the differential equation $dy/dt = ky$, k a constant, that satisfies the given conditions.

11. $k = 1.5$, $y(0) = 100$ $y(t) = 100e^{1.5t}$

12. $k = -0.5$, $y(0) = 200$ $y(t) = 200e^{-0.5t}$

13. $y(0) = 50$, $y(5) = 100$ $y(t) = 50e^{(0.2\ln 2)t}$

14. $y(0) = 60$, $y(10) = 30$ $y(t) = 60e^{-(0.1\ln 2)t}$

In Exercises 15–18, complete the table for an investment if interest is compounded continuously.

	Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
15.	1000	8.6		
16.	2000		15	
17.		5.25		2898.44
18.	1200			10,405.37

15. 8.06 yr doubling time; $13,197.14 in 30 yr **16.** 4.62% rate; $8000 in 30 yr **17.** $600 initially; 13.2 yr doubling time **18.** 7.2% rate; 9.63 yr doubling time

In Exercises 19 and 20, find the amount of time required for a $2000 investment to double if the annual interest rate r is compounded **(a)** annually, **(b)** monthly, **(c)** quarterly, and **(d)** continuously.

19. $r = 4.75\%$

20. $r = 8.25\%$ **(a)** 8.74 yr **(b)** 8.43 yr **(c)** 8.49 yr **(d)** 8.40 yr

21. ***Half-Life*** The radioactive decay of Sm-151 (an isotope of samarium) can be modeled by the differential equation $dy/dt = -0.0077y$, where t is measured in years. Find the half-life of Sm-151. 90 years.

22. ***Half-Life*** An isotope of neptunium (Np-240) has a half-life of 65 minutes. If the decay of Np-240 is modeled by the differential equation $dy/dt = -ky$, where t is measured in minutes, what is the decay constant k? $k = .01067$

23. ***Growth of Cholera Bacteria*** Suppose that the cholera bacteria in a colony grows unchecked according to the Law of Exponential Change. The colony starts with 1 bacterium and doubles in number every half hour.

(a) How many bacteria will the colony contain at the end of 24 h? 2.8×10^{14} bacteria

(b) **Writing to Learn** Use part (a) to explain why a person who feels well in the morning may be dangerously ill by evening even though, in an infected person, many bacteria are destroyed. The bacteria reproduce fast enough that even if many are destroyed, there are enough left to make the person sick.

24. ***Bacteria Growth*** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 h there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially? 1250 bacteria

25. ***Radon-222*** The decay equation for radon-222 gas is known to be $y = y_0e^{-0.18t}$, with t in days. About how long will it take the amount of radon in a sealed sample of air to decay to 90% of its original value? 0.585 days

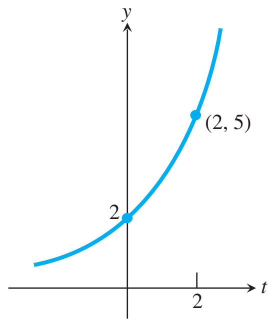
26. ***Polonium-210*** The number of radioactive atoms remaining after t days in a sample of polonium-210 that starts with y_0 radioactive atoms is $y = y_0e^{-0.005t}$.

(a) Find the element's half-life. 138.6 days

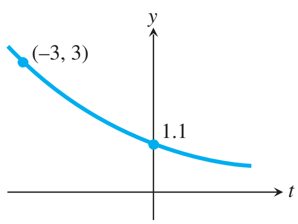
(b) Your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives have disintegrated. For about how many days after the sample arrives will you be able to use the polonium? 599 days

In Exercises 27 and 28, find the exponential function $y = y_0e^{kt}$ whose graph passes through the two points.

27. $y \approx 2e^{0.4581t}$



28. $y \approx 1.1e^{-0.3344t}$



29. *Mean Life of Radioactive Nuclei* Physicists using the radioactive decay equation $y = y_0e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon-222 nucleus is about $1/0.18 \approx 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in any sample will disintegrate within three mean lifetimes, that is, by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last. $y = y_0e^{-kt} = y_0e^{-k(3/k)} = y_0e^{-3} < 0.05y_0$

30. *Finding the Original Temperature of a Beam* An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°F. After 10 min, the beam warmed to 35°F and after another 10 min its temperature was 50°F. Use Newton's Law of Cooling to estimate the beam's initial temperature. 5°F

31. *Cooling Soup* Suppose that a cup of soup cooled from 90°C to 60°C in 10 min in a room whose temperature was 20°C. Use Newton's Law of Cooling to answer the following questions.

(a) How much longer would it take the soup to cool to 35°C? 17.53 minutes longer

(b) Instead of being left to stand in the room, the cup of 90°C soup is put into a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C? 13.26 minutes

32. *Cooling Silver* The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

(a) 15 minutes from now? 53.45° above room temperature

(b) 2 hours from now? 23.79° above room temperature

(c) When will the silver be 10°C above room temperature? 232.5 min or 3.9 hours

33. *Temperature Experiment* A temperature probe is removed from a cup of coffee and placed in water whose temperature (T_s) is 10°C. The data in Table 6.2 were collected over the next 30 sec with a CBL™ temperature probe.

Table 6.2 Experimental Data

Time (sec)	T (ºC)	$T - T_s$ (ºC)
2	80.47	70.47
5	69.39	59.39
10	49.66	39.66
15	35.26	25.26
20	28.15	18.15
25	23.56	13.56
30	20.62	10.62

(a) Find an exponential regression equation for the $(t, T - T_s)$ data. $T - T_s = 79.47(0.932^t)$

(b) Use the regression equation in part (a) to find a model for the (t, T) data. Superimpose the graph of the model on a scatter plot of the (t, T) data. See answer section.

(c) Estimate when the temperature probe will read 12°C. 52.5 seconds

(d) Estimate the coffee's temperature when the temperature probe was removed. 89.47°C

34. *A Very Cool Experiment* A temperature probe is removed from a cup of hot chocolate and placed in ice water (temperature $T_s = 0°C$). The data in Table 6.3 were collected over the next 30 seconds.

Table 6.3 Experimental Data

Time (sec)	Temperature (°C)
2	74.68
5	61.99
10	34.89
15	21.95
20	15.36
25	11.89
30	10.02

(a) Newton's Law of Cooling predicts that the difference between the probe temperature (T) and the surrounding temperature (T_s) is an exponential function of time, but in this case $T_s = 0$, so T is an exponential function of time.

(a) Writing to Learn Explain why temperature in this experiment can be modeled as an exponential function of time.

(b) Use exponential regression to find the best exponential model. Superimpose a graph of the model on a scatter plot of the (*time, temperature*) data. See answer section.

(c) Estimate when the probe will reach 5°C. At about 37 seconds.

(d) Estimate the temperature of the hot chocolate when the probe was removed. 79.96°C

35. *Dating Crater Lake* The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake? 6658 years

44. (a) $A(t) = A_0e^t$; It grows by a factor of e each year. **(c)** $(e - 1)$ times your initial amount, or $\approx$ 172% increase.

36. *Carbon-14 Dating Measurement Sensitivity* To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, answer the following questions about this hypothetical situation.

(a) A fossilized bone found in central Illinois in the year A.D. 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died. 12,571 B.C.

(b) Repeat part (a) assuming 18% instead of 17%. 12,101 B.C.

(c) Repeat part (a) assuming 16% instead of 17%. 13,070 B.C.

37. What is the half-life of a substance that decays to 1/3 of its original radioactive amount in 5 years? 3.15 years

38. A savings account earning compound interest triples in value in 10 years. How long will it take for the original investment to quadruple? 12.62 years

39. *The Inversion of Sugar* The processing of raw sugar has an "inversion" step that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 h, how much raw sugar will remain after another 14 h? 585.4 kg

40. *Oil Depletion* Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present level? 16.09 years

41. *Atmospheric Pressure* Earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 lb/in^2) and that the pressure at an altitude of 20 km is 90 millibars.

(a) Solve the initial value problem $p = 1013e^{-0.121h}$

Differential equation: $\dfrac{dp}{dh} = kp$,

Initial condition: $p = p_0$ when $h = 0$,

to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.

(b) What is the atmospheric pressure at $h = 50$ km? 2.383 millibars

(c) At what altitude does the pressure equal 900 millibars? 0.977 km

42. *First Order Chemical Reactions* In some chemical reactions the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when y is measured in grams and t is measured in hours. If there are 100 grams of a δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour? 54.88 grams

43. *Discharging Capacitor Voltage* Suppose that electricity is draining from a capacitor at a rate proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

(a) Solve this differential equation for V, using V_0 to denote the value of V when $t = 0$. $V = V_0e^{-t/40}$

(b) How long will it take the voltage to drop to 10% of its original value? 92.1 seconds

44. *John Napier's Answer* John Napier (1550–1617), the Scottish laird who invented logarithms, was the first person to answer the question, "What happens if you invest an amount of money at 100% yearly interest, compounded continuously?"

(a) Writing to Learn What does happen? Explain.

(b) How long does it take to triple your money? $\ln 3 \approx 1.1$ yr

(c) Writing to Learn How much can you earn in a year?

45. *Benjamin Franklin's Will* The Franklin Technical Institute of Boston owes its existence to a provision in a codicil to Benjamin Franklin's will. In part the codicil reads:

> I wish to be useful even after my Death, if possible, in forming and advancing other young men that may be serviceable to their Country in both Boston and Philadelphia. To this end I devote Two thousand Pounds Sterling, which I give, one thousand thereof to the Inhabitants of the Town of Boston in Massachusetts, and the other thousand to the inhabitants of the City of Philadelphia, in Trust and for the Uses, Interests and Purposes hereinafter mentioned and declared.

Franklin's plan was to lend money to young apprentices at 5% interest with the provision that each borrower should pay each year along

> … with the yearly Interest, one tenth part of the Principal, which sums of Principal and Interest shall be again let to fresh Borrowers. … If this plan is executed and succeeds as projected without interruption for one hundred Years, the Sum will then be one hundred and thirty-one thousand Pounds of which I would have the Managers of the Donation to the Inhabitants of the Town of Boston, then lay out at their discretion one hundred thousand Pounds in Public Works. … The remaining thirty-one thousand Pounds, I would have continued to be let out on Interest in the manner above directed for another hundred Years. … At the end of this second term if no unfortunate accident has prevented the operation the sum will be Four Millions and Sixty-one Thousand Pounds.

It was not always possible to find as many borrowers as Franklin had planned, but the managers of the trust did the best they could. At the end of 100 years from the receipt of the Franklin gift, in January 1894, the fund had grown from 1000 pounds to almost 90,000 pounds. In 100 years the original capital had multiplied about 90 times instead of the 131 times Franklin had imagined.

(a) What annual rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90? $\dfrac{\ln 90}{100} = 0.045$ or 4.5%

(b) In Benjamin Franklin's estimate that the original 1000 pounds would grow to 131,000 in 100 years, he was using an annual rate of 5% and compounding once each year. What rate of interest per year when compounded continuously for 100 years would multiply the original amount by 131? $\dfrac{\ln 131}{100} = 0.049$ or 4.9%

46. *Rules of 70 and 72* The rules state that it takes about $70/i$ or $72/i$ years for money to double at i percent, compounded continuously, using whichever of 70 or 72 is easier to divide by i.

(a) Show that it takes $t = (\ln 2)/r$ years for money to double if it is invested at annual interest rate r (in decimal form) compounded continuously. $2y_0 = y_0e^{rt} \Rightarrow t = \ln 2/r$

(b) Graph the functions See answer section.

$$y_1 = \frac{\ln 2}{r}, \quad y_2 = \frac{70}{100r}, \quad \text{and} \quad y_3 = \frac{72}{100r}$$

in the $[0, 0.1]$ by $[0, 100]$ viewing window.

(c) Writing to Learn Explain why these two rules of thumb for mental computation are reasonable. See answer section.

(d) Use the rules to estimate how long it takes to double money at 5% compounded continuously. 70/5 = 14 years or 72/5 = 14.4 years

(e) Invent a rule for estimating the number of years needed to triple your money. $108/(100r)$ (108 has more factors than 110.)

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

47. True or False If $dy/dx = ky$, then $y = e^{kx} + C$. Justify your answer. See page 361.

48. True or False The general solution to $dy/dt = 2y$ can be written in the form $y = C(3^{kt})$ for some constants C and k. Justify your answer. See page 361.

49. Multiple Choice A bank account earning continuously compounded interest doubles in value in 7.0 years. At the same interest rate, how long would it take the value of the account to triple? D

(A) 4.4 years **(B)** 9.8 years **(C)** 10.5 years
(D) 11.1 years **(E)** 21.0 years

50. Multiple Choice A sample of Ce-143 (an isotope of cerium) loses 99% of its radioactive matter in 199 hours. What is the half-life of Ce-143? C

(A) 4 hours **(B)** 6 hours **(C)** 30 hours
(D) 100.5 hours **(E)** 143 hours

51. Multiple Choice In which of the following models is dy/dt directly proportional to y? D

I. $y = e^{kt} + C$
II. $y = Ce^{kt}$
III. $y = 28^{kt}$

(A) I only **(B)** II only **(C)** I and II only
(D) II and III only **(E)** I, II, and III

52. Multiple Choice An apple pie comes out of the oven at 425°F and is placed on a counter in a 68°F room to cool. In 30 minutes it has cooled to 195°F. According to Newton's Law of Cooling, how many additional minutes must pass before it cools to 100°F? E

(A) 12.4 **(B)** 15.4 **(C)** 25.0 **(D)** 35.0 **(E)** 40.0

Explorations

53. *Resistance Proportional to Velocity* It is reasonable to assume that the air resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The resisting force on an object of mass m moving with velocity v is thus $-kv$ for some positive constant k.

(a) Use the law *Force* = *Mass* × *Acceleration* to show that the velocity of an object slowed by air resistance (and no other forces) satisfies the differential equation See page 361.

$$m\frac{dy}{dt} = -kv.$$

(b) Solve the differential equation to show that $v = v_0 e^{-(k/m)t}$, where v_0 is the velocity of the object at time $t = 0$. See page 361.

(c) If k is the same for two objects of different masses, which one will slow to half its starting velocity in the shortest time? Justify your answer. See page 361.

54. *Coasting to a Stop* Assume that the resistance encountered by a moving object is proportional to the object's velocity so that its velocity is $v = v_0e^{-(k/m)t}$.

(a) Integrate the velocity function with respect to t to obtain the distance function s. Assume that $s(0) = 0$ and show that

$$s(t) = \frac{v_0m}{k}\left(1 - e^{-(k/m)t}\right).$$

(b) Show that the total coasting distance traveled by the object as it coasts to a complete stop is v_0m/k. $\lim_{t\to\infty} s(t) = \frac{v_0m}{k}$

55. *Coasting to a Stop* Table 6.4 shows the distance s (meters) coasted on in-line skates in terms of time t (seconds) by Kelly Schmitzer. Find a model for her position in the form given in Exercise 54(a) and superimpose its graph on a scatter plot of the data. Her initial velocity was $v_0 = 0.80$ m/sec, her mass $m = 49.90$ kg (110 lb), and her total coasting distance was 1.32 m. See answer section.

Table 6.4 Kelly Schmitzer Skating Data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	1.5	0.89	3.1	1.30
0.1	0.07	1.7	0.97	3.3	1.31
0.3	0.22	1.9	1.05	3.5	1.32
0.5	0.36	2.1	1.11	3.7	1.32
0.7	0.49	2.3	1.17	3.9	1.32
0.9	0.60	2.5	1.22	4.1	1.32
1.1	0.71	2.7	1.25	4.3	1.32
1.3	0.81	2.9	1.28	4.5	1.32

Source: Valerie Sharrits, St. Francis de Sales H.S., Columbus, OH.

56. *Coasting to a Stop* Table 6.5 shows the distance s (meters) coasted on in-line skates in t seconds by Johnathon Krueger. Find a model for his position in the form given in Exercise 54(a) and superimpose its graph on a scatter plot of the data. His initial velocity was $v_0 = 0.86$ m/sec, his mass $m = 30.84$ kg (he weighed 68 lb), and his total coasting distance 0.97 m. See answer section.

Table 6.5 Johnathon Krueger Skating Data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	0.93	0.61	1.86	0.93
0.13	0.08	1.06	0.68	2.00	0.94
0.27	0.19	1.20	0.74	2.13	0.95
0.40	0.28	1.33	0.79	2.26	0.96
0.53	0.36	1.46	0.83	2.39	0.96
0.67	0.45	1.60	0.87	2.53	0.97
0.80	0.53	1.73	0.90	2.66	0.97

Source: Valerie Sharrits, St. Francis de Sales H.S., Columbus, OH.

Extending the Ideas

57. *Continuously Compounded Interest*

(a) Use tables to give a numerical argument that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Support your argument graphically.

(b) For several different values of r, give numerical and graphical evidence that

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = e^r.$$

(c) Writing to Learn Explain why compounding interest over smaller and smaller periods of time leads to the concept of interest compounded continuously.

58. *Skydiving* If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity $v(t)$ is modeled by the initial value problem

Differential equation: $m\dfrac{dv}{dt} = mg - kv^2,$

Initial condition: $v(0) = 0,$

where t represents time in seconds, g is the acceleration due to gravity, and k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that variation in the air's density will not affect the outcome.)

(a) Show that the function

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{e^{at}-e^{-at}}{e^{at}+e^{-at}},$$

where $a = \sqrt{gk/m}$, is a solution of the initial value problem.

(b) Find the body's limiting velocity, $\lim_{t\to\infty} v(t)$. $\sqrt{mg/k}$

(c) For a 160-lb skydiver ($mg = 160$), and with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity in feet per second? in miles per hour?

179 ft/sec ≈ 122 mi/hr

Skydivers can vary their limiting velocities by changing the amount of body area opposing the fall. Their velocities can vary from 94 to 321 miles per hour.

Answers:

47. False. The correct solution is $|y| = e^{kx+C}$, which can be written (with a new C) as $y = Ce^{kx}$.

48. True. The differential equation is solved by an exponential equation that can be written in any base. Note that $Ce^{2t} = C(3^{kt})$ when $k = 2/(\ln 3)$.

53. (a) Since acceleration is $\dfrac{dv}{dt}$, we have Force $= m\dfrac{dv}{dt} = -kv$.

(b) From $m\dfrac{dv}{dt} = -kv$ we get $\dfrac{dv}{dt} = -\dfrac{k}{m}v$, which is the differential equation for exponential growth modeled by $v = Ce^{-(k/m)t}$. Since $v = v_0$ at $t = 0$, it follows that $C = v_0$.

(c) In each case, we would solve $2 = e^{-(k/m)t}$. If k is constant, an increase in m would require an increase in t. The object of larger mass takes longer to slow down. Alternatively, one can consider the equation $\dfrac{dv}{dt} = -\dfrac{k}{m}v$ to see that v changes more slowly for larger values of m.

6.5 Logistic Growth

What you'll learn about

- How Populations Grow
- Partial Fractions
- The Logistic Differential Equation
- Logistic Growth Models

. . . and why

Populations in the real world tend to grow logistically over extended periods of time.

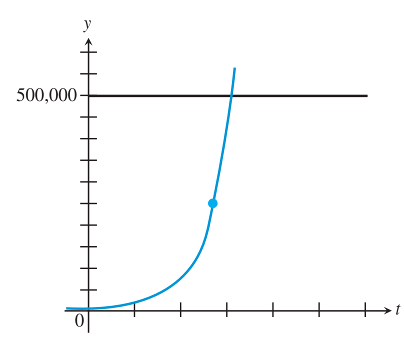
How Populations Grow

In Section 6.4 we showed that when the rate of change of a population is directly proportional to the size of the population, the population grows exponentially. This seems like a reasonable model for population growth in the short term, but populations in nature cannot sustain exponential growth for very long. Available food, habitat, and living space are just a few of the constraints that will eventually impose limits on the growth of any real-world population.

EXPLORATION 1 **Exponential Growth Revisited**

Almost any algebra book will include a problem like this: A culture of bacteria in a Petri dish is doubling every hour. If there are 100 bacteria at time $t = 0$, how many bacteria will there be in 12 hours?

1. Answer the algebra question.
2. Suppose a textbook editor, seeking to add a little unit conversion to the problem to satisfy a reviewer, changes "12 hours" to "12 days" in the second edition of the textbook. What is the answer to the revised question?
3. Is the new answer reasonable? (*Hint:* It has been estimated that there are about 10^{79} atoms in the entire universe.)
4. Suppose the maximal sustainable population of bacteria in this Petri dish is 500,000 bacteria. How many hours will it take the bacteria to reach this population if the exponential model continues to hold?
5. The graph below shows what the graph of the population would look like if it were to remain exponential until hitting 500,000. Draw a more reasonable graph that shows how the population might approach 500,000 after growing exponentially up to the marked point.



You might recall that we introduced *logistic* curves in Section 4.3 to illustrate points of inflection. Logistic growth, which starts off exponentially and then changes concavity to approach a maximal sustainable population, is a better model for real-world populations, for all the reasons mentioned above.

Partial Fractions

Before we introduce the differential equation that describes logistic growth, we need to review a bit of algebra that is needed to solve it.

Partial Fraction Decomposition with Distinct Linear Denominators

If $f(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials with the degree of P less than the degree of Q, and if $Q(x)$ can be written as a product of distinct linear factors, then $f(x)$ can be written as a sum of rational functions with distinct linear denominators.

We will illustrate this principle with examples.

A Question of Degree

Note that the technique of partial fractions only applies to rational functions of the form

$$\frac{P(x)}{Q(x)}$$

where P and Q are polynomials with the degree of P less than the degree of Q. Such a fraction is called *proper*. Example 2 will show you how to handle an *improper* fraction.

EXAMPLE 1 Finding a Partial Fraction Decomposition

Write the function $f(x) = \dfrac{x - 13}{2x^2 - 7x + 3}$ as a sum of rational functions with linear denominators.

SOLUTION

Since $f(x) = \dfrac{x - 13}{(2x - 1)(x - 3)}$, we will find numbers A and B so that

$f(x) = \dfrac{A}{2x - 1} + \dfrac{B}{x - 3}$.

Note that $\dfrac{A}{2x - 1} + \dfrac{B}{x - 3} = \dfrac{A(x - 3) + B(2x - 1)}{(2x - 1)(x - 3)}$, so it follows that

$$A(x - 3) + B(2x - 1) = x - 13. \qquad (1)$$

Setting $x = 3$ in equation (1), we get

$$A(0) + B(5) = -10, \text{ so } B = -2.$$

Setting $x = \dfrac{1}{2}$ in equation (1), we get

$$A\left(-\frac{5}{2}\right) + B(0) = -\frac{25}{2}, \text{ so } A = 5.$$

Therefore $f(x) = \dfrac{x - 13}{(2x - 1)(x - 3)} = \dfrac{5}{2x - 1} - \dfrac{2}{x - 3}$.

Now try Exercise 3.

A Little on the Heaviside

The substitution technique used to find A and B in Example 1 (and in subsequent examples) is often called the Heaviside Method after English engineer Oliver Heaviside (1850–1925).

You might already have guessed that partial fraction decomposition can be of great value when antidifferentiating rational functions.

EXAMPLE 2 Antidifferentiating with Partial Fractions

Find $\displaystyle\int \frac{3x^4 + 1}{x^2 - 1}\, dx$.

SOLUTION

First we note that the degree of the denominator is not less than the degree of the numerator. We use the division algorithm to find the quotient and remainder:

$$\begin{array}{r|l} & 3x^2 + 3 \\ x^2 - 1 & 3x^4 \qquad\quad + 1 \\ & \underline{3x^4 - 3x^2} \\ & \qquad 3x^2 + 1 \\ & \qquad \underline{3x^2 - 3} \\ & \qquad\qquad 4 \end{array}$$

continued

Thus

$$\int \frac{3x^4+1}{x^2-1}\,dx = \int \left(3x^2+3+\frac{4}{x^2-1}\right)dx$$
$$= x^3+3x+\int \frac{4}{(x-1)(x+1)}\,dx$$
$$= x^3+3x+\int \left(\frac{A}{x-1}+\frac{B}{x+1}\right)dx.$$

We know that $A(x+1)+B(x-1)=4$.

Setting $x=1$,

$$A(2)+B(0)=4, \text{ so } A=2.$$

Setting $x=-1$,

$$A(0)+B(-2)=4, \text{ so } B=-2.$$

Thus

$$\int \frac{3x^4+1}{x^2-1}\,dx = x^3+3x+\int \left(\frac{2}{x-1}+\frac{-2}{x+1}\right)dx$$
$$= x^3+3x+2\ln|x-1|-2\ln|x+1|+C$$
$$= x^3+3x+2\ln\left|\frac{x-1}{x+1}\right|+C.$$

Now try Exercise 7.

EXAMPLE 3 Finding Three Partial Fractions

This example will be our most laborious problem.

Find the general solution to $\dfrac{dy}{dx} = \dfrac{6x^2-8x-4}{(x^2-4)(x-1)}$.

SOLUTION

$$y = \int \frac{6x^2-8x-4}{(x-2)(x+2)(x-1)}\,dx = \int \left(\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x-1}\right)dx.$$

We know that $A(x+2)(x-1)+B(x-2)(x-1)+C(x-2)(x+2)=6x^2-8x-4$.

Setting $x=2$:

$$A(4)(1)+B(0)+C(0)=4, \text{ so } A=1.$$

Setting $x=-2$:

$$A(0)+B(-4)(-3)+C(0)=36, \text{ so } B=3.$$

Setting $x=1$,

$$A(0)+B(0)+C(-1)(3)=-6, \text{ so } C=2.$$

Thus

$$\int \frac{6x^2-8x-4}{(x-2)(x+2)(x-1)}\,dx = \int \left(\frac{1}{x-2}+\frac{3}{x+2}+\frac{2}{x-1}\right)dx$$
$$= \ln|x-2|+3\ln|x+2|+2\ln|x-1|+C$$
$$= \ln\left(|x-2||x+2|^3\,|x-1|^2\right)+C.$$

Now try Exercise 17.

The technique of partial fractions can actually be extended to apply to all rational functions, but the method has to be adapted slightly if there are repeated linear factors or irreducible quadratic factors in the denominator. Both of these cases lead to partial fractions with quadratic denominators, and we will not deal with them in this book.

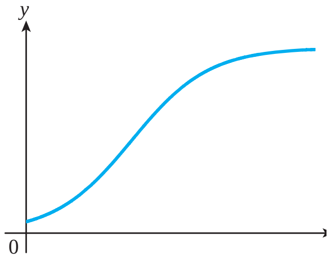


Figure 6.13 A logistic curve.

The Logistic Differential Equation

Now consider the case of a population P with a growth curve as a function of time that begins increasing and concave up (as in exponential growth), then turns increasing and concave down as it approaches the carrying capacity of its habitat. A **logistic curve,** like the one shown in Figure 6.13, has the shape to model this growth.

We have seen that the exponential growth at the beginning can be modeled by the differential equation

$$\frac{dP}{dt} = kP \text{ for some } k > 0.$$

If we want the growth rate to approach zero as P approaches a maximal **carrying capacity** M, we can introduce a limiting factor of $M - P$:

$$\frac{dP}{dt} = kP(M - P)$$

This is the **logistic differential equation.** Before we find its general solution, let us see how much we can learn about logistic growth just by studying the differential equation itself.

EXPLORATION 2 Learning from the Differential Equation

Consider a (positive) population P that satisfies $dP/dt = kP(M - P)$, where k and M are positive constants.

1. For what values of P will the growth rate dP/dt be close to zero?
2. As a function of P, $y = kP(M - P)$ has a graph that is an upside-down parabola. What is the value of P at the vertex of that parabola?
3. Use the answer to part (2) to explain why the growth rate is maximized when the population reaches half the carrying capacity.
4. If the initial population is less than M, is the initial growth rate positive or negative?
5. If the initial population is greater than M, is the initial growth rate positive or negative?
6. If the initial population equals M, what is the initial growth rate?
7. What is $\lim_{t\to\infty} P(t)$? Does it depend on the initial population?

You can use the results of Exploration 2 in the following example.

EXAMPLE 4

The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $dP/dt = 0.008P(100 - P)$, where t is measured in years.

(a) What is the carrying capacity for bears in this wildlife preserve?

(b) What is the bear population when the population is growing the fastest?

(c) What is the rate of change of the population when it is growing the fastest?

continued

SOLUTION

(a) The carrying capacity is 100 bears.

(b) The bear population is growing the fastest when it is half the carrying capacity, 50 bears.

(c) When $P = 50$, $dP/dt = 0.008(50)(100 - 50) = 20$ bears per year. Although the derivative represents the instantaneous growth rate, it is reasonable to say that the population grows by about 20 bears that year.

Now try Exercise 25.

In this next example we will find the solution to a logistic differential equation with an initial condition.

EXAMPLE 5 Tracking a Moose Population

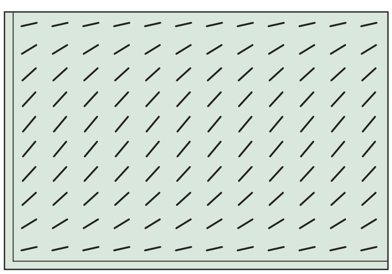
In 1985 and 1987, the Michigan Department of Natural Resources airlifted 61 moose from Algonquin Park, Ontario to Marquette County in the Upper Peninsula. It was originally hoped that the population P would reach carrying capacity in about 25 years with a growth rate of

$$\frac{dP}{dt} = 0.0003P(1000 - P).$$

(a) According to the model, what is the carrying capacity?

(b) With a calculator, generate a slope field for the differential equation.

(c) Solve the differential equation with the initial condition $P(0) = 61$ and show that it conforms to the slope field.



[0, 25] by [0, 1000]

Figure 6.14 The slope field for the moose differential equation in Example 5.

SOLUTION

(a) The carrying capacity is 1000 moose.

(b) The slope field is shown in Figure 6.14. Since the population approaches a horizontal asymptote at 1000 in about 25 years, we use the window [0, 25] by [0, 1000].

(c) After separating the variables, we encounter an antiderivative to be found using partial fractions.

$$\frac{dP}{P(1000 - P)} = 0.0003\, dt$$

$$\int \frac{1}{P(1000 - P)}\, dP = \int 0.0003\, dt$$

$$\int \left(\frac{A}{P} + \frac{B}{1000 - P}\right) dP = \int 0.0003\, dt$$

We know that $A(1000 - P) + B(P) = 1$.

Setting $P = 0$: $A(1000) + B(0) = 1$, so $A = 0.001$.

Setting $P = 1000$: $A(0) + B(1000) = 1$, so $B = 0.001$.

continued

$$\int\left(\frac{0.001}{P}+\frac{0.001}{1000-P}\right)dP=\int 0.0003\,dt$$

$$\int\left(\frac{1}{P}+\frac{1}{1000-P}\right)dP=\int 0.3\,dt$$ Multiplied both integrals by 1000.

$$\ln P-\ln(1000-P)=0.3t+C$$

$$\ln(1000-P)-\ln P=-0.3t-C$$

$$\ln\left(\frac{1000-P}{P}\right)=-0.3t-C$$

$$\frac{1000}{P}-1=e^{-0.3t-C}$$

$$\frac{1000}{P}=1+e^{-0.3t}\,e^{-C}$$

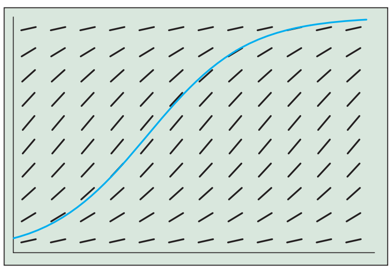
Setting $P=61$ and $t=0$, we find that $e^{-C}\approx 15.393$. Thus

$$\frac{1000}{P}=1+15.393e^{-0.3t}$$

$$P=\frac{1000}{1+15.393e^{-0.3t}}.$$

The graph conforms nicely to the slope field, as shown in Figure 6.15.

Now try Exercise 29.



[0, 25] by [0, 1000]

Figure 6.15 The particular solution

$$P=\frac{1000}{1+15.393e^{-0.3t}}$$

conforms nicely to the slope field for $dP/dt=0.0003P\,(1000-P)$. (Example 5)

Logistic Growth Models

We could solve many more logistic differential equations and the algebra would look the same every time. In fact, it is almost as simple to solve the equation using letters for all the constants, thereby arriving at a general formula. In Exercise 35 we will ask you to verify the result in the box below.

The General Logistic Formula

The solution of the general logistic differential equation

$$\frac{dP}{dt}=kP(M-P)$$

is

$$P=\frac{M}{1+Ae^{-(Mk)t}}$$

where A is a constant determined by an appropriate initial condition. The **carrying capacity** M and the **growth constant** k are positive constants.

We introduced logistic regression in Example 6 of Section 4.3. We close this section with another logistic regression example that makes use of our additional understanding.

EXAMPLE 6 Using Logistic Regression

Table 6.6 shows the population of Aurora, CO for selected years between 1950 and 2003.

Table 6.6 Population of Aurora, CO

Years after 1950	Population
0	11,421
20	74,974
30	158,588
40	222,103
50	275,923
53	290,418

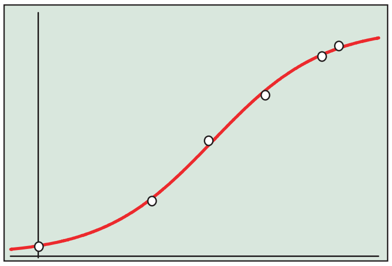
Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a logistic curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what will the Aurora population approach in the long run?

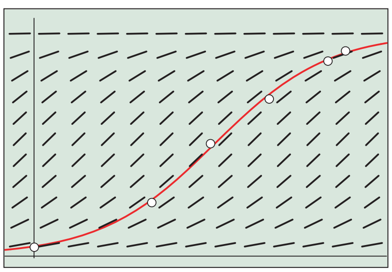
(c) Based on the regression equation, when will the population of Aurora first exceed 300,000 people?

(d) Write a logistic differential equation in the form $dP/dt = kP(M - P)$ that models the growth of the Aurora data in Table 6.6.



[–5, 60] by [–3600, 337000]

Figure 6.16 The logistic regression curve fitted to the data for population growth in Aurora, CO from 1950 to 2003. (Example 6)



[–5, 60] by [–3600, 337000]

Figure 6.17 The slope field for the differential equation derived from the regression curve fits the data and the regression curve nicely. (Example 6)

SOLUTION

(a) The regression equation is

$$P = \frac{316440.7}{1 + 23.577e^{-0.1026t}}.$$

The graph is shown superimposed on the scatter plot in Figure 6.16. The fit is almost perfect.

(b) Approximately 316,441 people. (The carrying capacity is the numerator of the regression equation.)

(c) Set

$$\frac{316440.7}{1 + 23.577e^{-0.1026t}} = 300{,}000.$$

The regression line crosses the 300,000 mark sometime in the 59$^{\text{th}}$ year, that is, in 2009.

(d) We see from the regression equation that $M = 316440.7$ and $Mk = 0.1026$. Therefore $k \approx 3.24 \times 10^{-7}$. The logistic growth model is

$$\frac{dP}{dt} = (3.24 \times 10^{-7})\, P(316440.7 - P).$$

Figure 6.17 shows the slope field for this differential equation superimposed on the scatter plot and the regression equation. ***Now try Exercise 37.***

Graphing Calculator Logistics

Unfortunately, some graphing calculators allow for a *vertical shift* when fitting a logistic curve to a set of data points. The regression equation for such a curve would have the form

$$y = \frac{c}{1 + ae^{-bx}} + d.$$

While the curve might fit the data better, this function cannot be a solution to the logistic differential equation if d is not zero. Since our definition of a logistic function begins with the differential equation, we will consistently use only logistic regression equations of the form

$$y = \frac{c}{1 + ae^{-bx}}.$$

We caution readers once again not to assume that logistic models work perfectly in all real-world, population-growth problems; there are too many unpredictable variables that can and will change the growth conditions over time.

Quick Review 6.5 *(For help, go to Sections 2.2 and 2.3.)*

In Exercises 1–4, use the polynomial division algorithm (as in Example 2 of this section) to write the rational function in the form $Q(x) + \dfrac{R(x)}{D(x)}$, where the degree of R is less than the degree of D.

1. $\dfrac{x^2}{x-1}$ $\quad x + 1 + \dfrac{1}{x-1}$

2. $\dfrac{x^2}{x^2-4}$ $\quad 1 + \dfrac{4}{x^2-4}$

3. $\dfrac{x^2+x+1}{x^2+x-2}$ $\quad 1 + \dfrac{3}{x^2+x-2}$

4. $\dfrac{x^3-5}{x^2-1}$ $\quad x + \dfrac{x-5}{x^2-1}$

In Exercises 5–10, let $f(x) = \dfrac{60}{1 + 5e^{-0.1x}}$.

5. Find where f is continuous. $(-\infty, \infty)$

6. Find $\lim_{x\to\infty} f(x)$. 60

7. Find $\lim_{x\to-\infty} f(x)$. 0

8. Find the y-intercept of the graph of f. 10

9. Find all horizontal asymptotes of the graph of f. $y = 0, y = 60$

10. Draw the graph of $y = f(x)$. See answer section.

Section 6.5 Exercises

In Exercises 1–4, find the values of A and B that complete the partial fraction decomposition.

1. $\dfrac{x-12}{x^2-4x} = \dfrac{A}{x} + \dfrac{B}{x-4}$ $\quad A = 3, B = -2$

2. $\dfrac{2x+16}{x^2+x-6} = \dfrac{A}{x+3} + \dfrac{B}{x-2}$ $\quad A = -2, B = 4$

3. $\dfrac{16-x}{x^2+3x-10} = \dfrac{A}{x-2} + \dfrac{B}{x+5}$ $\quad A = 2, B = -3$

4. $\dfrac{3}{x^2-9} = \dfrac{A}{x-3} + \dfrac{B}{x+3}$ $\quad A = 1/2, B = -1/2$

In Exercises 5–14, evaluate the integral.

5. $\displaystyle\int \frac{x-12}{x^2-4x}\,dx$ $\quad \ln \dfrac{|x|^3}{(x-4)^2} + C$

6. $\displaystyle\int \frac{2x+16}{x^2+x-6}\,dx$ $\quad \ln \dfrac{(x-2)^4}{(x+3)^2} + C$

7. $\displaystyle\int \frac{2x^3}{x^2-4}\,dx$ $\quad x^2 + \ln(x^2-4)^4 + C$

8. $\displaystyle\int \frac{x^2-6}{x^2-9}\,dx$ $\quad x + \ln\sqrt{\left|\dfrac{x-3}{x+3}\right|} + C$

9. $\displaystyle\int \frac{2\,dx}{x^2+1}$ $\quad 2\tan^{-1} x + C$

10. $\displaystyle\int \frac{3\,dx}{x^2+9}$ $\quad \tan^{-1}\left(\dfrac{x}{3}\right) + C$

11. $\displaystyle\int \frac{7\,dx}{2x^2-5x-3}$ $\quad \ln\left|\dfrac{x-3}{2x+1}\right| + C$

12. $\displaystyle\int \frac{1-3x}{3x^2-5x+2}\,dx$ $\quad \ln\dfrac{|3x+2|}{(x-1)^2} + C$

13. $\displaystyle\int \frac{8x-7}{2x^2-x-3}\,dx$ $\quad \ln(|x+1|^3|2x-3|) + C$

14. $\displaystyle\int \frac{5x+14}{x^2+7x}\,dx$ $\quad \ln(x^2|x+7|^3) + C$

In Exercises 15–18, solve the differential equation.

15. $\dfrac{dy}{dx} = \dfrac{2x-6}{x^2-2x}$ See page 371.

16. $\dfrac{du}{dx} = \dfrac{2}{x^2-1}$ See page 371.

17. $F'(x) = \dfrac{2}{x^3-x}$ See page 371.

18. $G'(t) = \dfrac{2t^3}{t^3-t}$ See page 371.

In Exercises 19–22, find the integral *without* using the technique of partial fractions.

19. $\displaystyle\int \frac{2x}{x^2-4}\,dx$ $\quad \ln|x^2-4| + C$

20. $\displaystyle\int \frac{4x-3}{2x^2-3x+1}\,dx$ $\quad \ln|2x^2-3x+1| + C$

21. $\displaystyle\int \frac{x^2+x-1}{x^2-x}\,dx$ $\quad x + \ln|x^2-x| + C$

22. $\displaystyle\int \frac{2x^3}{x^2-1}\,dx$ $\quad x^2 + \ln|x^2-1| + C$

In Exercises 23–26, the logistic equation describes the growth of a population P, where t is measured in years. In each case, find **(a)** the carrying capacity of the population, **(b)** the size of the population when it is growing the fastest, and **(c)** the rate at which the population is growing when it is growing the fastest.

23. $\dfrac{dP}{dt} = 0.006P(200 - P)$ See page 371.

24. $\dfrac{dP}{dt} = 0.0008P(700 - P)$ See page 371.

25. $\dfrac{dP}{dt} = 0.0002P(1200 - P)$ See page 371.

26. $\dfrac{dP}{dt} = 10^{-5}P(5000 - P)$ See page 371.

In Exercises 27–30, solve the initial value problem using partial fractions. Use a graphing utility to generate a slope field for the differential equation and verify that the solution conforms to the slope field.

27. $\dfrac{dP}{dt} = 0.006P(200 - P)$ and $P = 8$ when $t = 0$.

28. $\dfrac{dP}{dt} = 0.0008P(700 - P)$ and $P = 10$ when $t = 0$.

29. $\dfrac{dP}{dt} = 0.0002P(1200 - P)$ and $P = 20$ when $t = 0$.

30. $\dfrac{dP}{dt} = 10^{-5}P(5000 - P)$ and $P = 50$ when $t = 0$.

In Exercises 31 and 32, a population function is given.

(a) Show that the function is a solution of a logistic differential equation. Identify k and the carrying capacity.

(b) Writing to Learn Estimate $P(0)$. Explain its meaning in the context of the problem.

31. ***Rabbit Population*** A population of rabbits is given by the formula **(a)** $k = 0.7; M = 1000$ **(b)** $P(0) \approx 8$; Initially there are 8 rabbits.

$$P(t) = \frac{1000}{1 + e^{4.8-0.7t}},$$

where t is the number of months after a few rabbits are released.

32. ***Spread of Measles*** The number of students infected by measles in a certain school is given by the formula

$$P(t) = \frac{200}{1 + e^{5.3-t}},$$

where t is the number of days after students are first exposed to an infected student. **(a)** $k = 1; M = 200$ **(b)** $P(0) \approx 1$; Initially 1 student has the measles.

33. ***Guppy Population*** A 2000-gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0015P(150 - P),$$

where time t is in weeks. $P(t) = \dfrac{150}{1 + 24e^{-0.225t}}$

(a) Find a formula for the guppy population in terms of t.

(b) How long will it take for the guppy population to be 100? 125? About 17.21 weeks; 21.28 weeks

34. ***Gorilla Population*** A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0004P(250 - P),$$

where time t is in years. $P(t) \approx \dfrac{250}{1 + 7.9286e^{-0.1t}}$

(a) Find a formula for the gorilla population in terms of t.

(b) How long will it take for the gorilla population to reach the carrying capacity of the preserve? 83 yrs to reach 249.5 ≈ 250

35. ***Logistic Differential Equation*** Show that the solution of the differential equation

$$\frac{dP}{dt} = kP(M - P) \quad \text{is} \quad P = \frac{M}{1 + Ae^{-Mkt}},$$

where A is a constant determined by an appropriate initial condition.

36. ***Limited Growth Equation*** Another differential equation that models limited growth of a population P in an environment with carrying capacity M is $dP/dt = k(M - P)$ (where $k > 0$ and $M > 0$).

(a) Show that $P = M - Ae^{-kt}$, where A is a constant determined by an appropriate initial condition.

(b) What is $\lim_{t\to\infty} P(t)$? M

(c) For what time $t \geq 0$ is the population growing the fastest? When $t = 0$.

(d) **Writing to Learn** How does the growth curve in this model differ from the growth curve in the logistic model? See answer section.

37. ***Population Growth*** Table 6.7 shows the population of Laredo, Texas for selected years between 1950 and 2003.

Table 6.7 Population of Laredo, TX

Years after 1950	Population
0	10,571
20	81,437
30	138,857
40	180,650
50	215,794
53	218,027

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

40. True. The graph will be a logistic curve with $\lim_{t\to\infty} p(t) = 100$ and $\lim_{t\to\infty} P(t) = 0$.

(a) Use logistic regression to find a curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what number will the Laredo population approach in the long run?

(c) Based on the regression equation, when will the Laredo population first exceed 225,000 people?

(d) Write a logistic differential equation in the form $dP/dt = kP(M - P)$ that models the growth of the Laredo data in Table 6.7.

38. ***Population Growth*** Table 6.8 shows the population of Virginia Beach, VA for selected years between 1950 and 2003.

Table 6.8 Population of Virginia Beach

Years after 1950	Population
0	5,390
20	172,106
30	262,199
40	393,069
50	425,257
53	439,467

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what number will the Virginia Beach population approach in the long run?

(c) Based on the regression equation, when will the Virginia Beach population first exceed 450,000 people?

(d) Write a logistic differential equation in the form $dP/dt = kP(M - P)$ that models the growth of the Virginia Beach data in Table 6.8.

Standardized Test Questions

You should solve the following problems without using a graphing calculator.

39. **True or False** For small values of t, the solution to logistic differential equation $dP/dt = kP(100 - P)$ that passes through the point (0, 10) resembles the solution to the differential equation $dP/dt = kP$ that passes through the point (0, 10). Justify your answer. False. It does look exponential, but it resembles the solution to $dP/dt = kP(100 - 10) = (90k)P$.

40. **True or False** The graph of any solution to the differential equation $dP/dt = kP(100 - P)$ has asymptotes $y = 0$ and $y = 100$. Justify your answer. See above.

41. **Multiple Choice** The spread of a disease through a community can be modeled with the logistic equation

$$\frac{dy}{dt} = \frac{600}{1 + 59e^{-0.1t}},$$

where y is the number of people infected after t days. How many people are infected when the disease is spreading the fastest? D

(A) 10 **(B)** 59 **(C)** 60 **(D)** 300 **(E)** 600

42. Multiple Choice The spread of a disease through a community can be modeled with the logistic equation

$$\frac{dy}{dt} = \frac{0.9}{1 + 45e^{-0.15t}},$$

where y is the proportion of people infected after t days. According to the model, what percentage of the people in the community will not become infected? B

(A) 2% **(B)** 10% **(C)** 15% **(D)** 45% **(E)** 90%

43. Multiple Choice $\int_2^3 \frac{3}{(x-1)(x+2)}\,dx =$ D

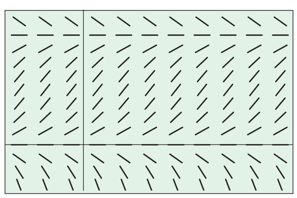
(A) $-\frac{33}{20}$ **(B)** $-\frac{9}{20}$ **(C)** $\ln\left(\frac{5}{2}\right)$ **(D)** $\ln\left(\frac{8}{5}\right)$ **(E)** $\ln\left(\frac{2}{5}\right)$

44. Multiple Choice Which of the following differential equations would produce the slope field shown below? B

(A) $\frac{dy}{dx} = 0.01x(120 - x)$ **(B)** $\frac{dy}{dx} = 0.01y(120 - y)$

(C) $\frac{dy}{dx} = 0.01y(100 - x)$ **(D)** $\frac{dy}{dx} = \frac{120}{1 + 60e^{-1.2x}}$

(E) $\frac{dy}{dx} = \frac{120}{1 + 60e^{-1.2y}}$



[–3, 8] by [–50, 150]

Explorations

45. *Extinct Populations* One theory states that if the size of a population falls below a minimum m, the population will become extinct. This condition leads to the *extended* logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$
$$= \frac{k}{M}(M - P)(P - m),$$

with $k > 0$ the proportionality constant and M the population maximum.

(a) Show that dP/dt is positive for $m < P < M$ and negative if $P < m$ or $P > M$.

(b) Let $m = 100$, $M = 1200$, and assume that $m < P < M$. Show that the differential equation can be rewritten in the form

$$\left[\frac{1}{1200 - P} + \frac{1}{P - 100}\right]\frac{dP}{dt} = \frac{11}{12}k.$$

Use a procedure similar to that used in Example 5 in Section 6.5 to solve this differential equation.

(c) Find the solution to part (b) that satisfies $P(0) = 300$.

(d) Superimpose the graph of the solution in part (c) with $k = 0.1$ on a slope field of the differential equation.

(e) Solve the general extended differential equation with the restriction $m < P < M$.

46. *Integral Tables* Antiderivatives of various generic functions can be found as formulas in *integral tables*. See if you can derive the formulas that would appear in an integral table for the following functions. (Here, a is an arbitrary constant.) See below.

(a) $\int \frac{dx}{a^2 + x^2}$ **(b)** $\int \frac{dx}{a^2 - x^2}$ **(c)** $\int \frac{dx}{(a + x)^2}$

Extending the Ideas

47. *Partial Fractions with Repeated Linear Factors*

If

$$f(x) = \frac{P(x)}{(x - r)^m}$$

is a rational function with the degree of P less than m, then the partial fraction decomposition of f is

$$f(x) = \frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

For example,

$$\frac{4x}{(x - 2)^2} = \frac{4}{x - 2} + \frac{8}{(x - 2)^2}.$$

Use partial fractions to find the following integrals:

(a) $\int \frac{5x}{(x + 3)^2}\,dx$ $5\ln|x + 3| + \frac{15}{x + 3} + C$

(b) $\int \frac{5x}{(x + 3)^3}\,dx$ (*Hint:* Use part (a).) $-\frac{5}{x + 3} + \frac{15}{2(x + 3)^2} + C$

48. *More on Repeated Linear Factors* The Heaviside Method is not very effective at finding the unknown numerators for partial fraction decompositions with repeated linear factors, but here is another way to find them. See answer section.

(a) If $\frac{x^2 + 3x + 5}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3}$, show that $A(x - 1)^2 + B(x - 1) + C = x^2 + 3x + 5$.

(b) Expand and equate coefficients of like terms to show that $A = 1$, $-2A + B = 3$, and $A - B + C = 5$. Then find A, B, and C.

(c) Use partial fractions to evaluate $\int \frac{x^2 + 3x + 5}{(x - 1)^3}\,dx$.

Answers:

15. $y = \ln\left|\frac{x^3}{x - 2}\right| + C$ **16.** $u = \ln\left|\frac{x - 1}{x + 1}\right| + C$

17. $F(x) = \ln\frac{|x^2 - 1|}{x^2} + C$ **18.** $G(x) = 2x + \ln\left|\frac{x - 1}{x + 1}\right| + C$

23. (a) 200 individuals **(b)** 100 individuals **(c)** 60 individuals per year
24. (a) 700 individuals **(b)** 350 individuals **(c)** 98 individuals per year
25. (a) 1200 individuals **(b)** 600 individuals **(c)** 72 individuals per year
26. (a) 5000 individuals **(b)** 2500 individuals **(c)** 62.5 individuals per year

46. (a) $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ **(b)** $\frac{1}{2a}\ln\left|\frac{x + a}{x - a}\right| + C$ **(c)** $-\frac{1}{x + a} + C$

Quick Quiz: Sections 6.4 and 6.5

You may use a graphing calculator to solve the following problems.

1. Multiple Choice The rate at which acreage is being consumed by a plot of kudzu is proportional to the number of acres already consumed at time t. If there are 2 acres consumed when $t = 1$ and 3 acres consumed when $t = 5$, how many acres will be consumed when $t = 8$? C

(A) 3.750 **(B)** 4.000 **(C)** 4.066 **(D)** 4.132 **(E)** 4.600

2. Multiple Choice Let $F(x)$ be an antiderivative of $\cos(x^2)$. If $F(1) = 0$, then $F(5) =$ C

(A) -0.099 **(B)** -0.153 **(C)** -0.293 **(D)** -0.992 **(E)** -1.833

3. Multiple Choice $\displaystyle\int \frac{dx}{(x-1)(x+3)} =$ A

(A) $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$ **(B)** $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right| + C$

(C) $\frac{1}{2}\ln|(x-1)(x+3)| + C$ **(D)** $\frac{1}{2}\ln\left|\frac{2x+2}{(x-1)(x+3)}\right| + C$

(E) $\ln|(x-1)(x+3)| + C$

4. Free Response A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{10}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$? 10

(b) If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$? 10

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{10}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t\to\infty} Y(t)$?

4. (c) Separate the variables.

$$\frac{dY}{Y} = \frac{1}{5}\left(1 - \frac{t}{10}\right)dt$$

$$\ln Y = \frac{t}{5} - \frac{t^2}{100} + C_1$$

$$Y = Ce^{t/5 - t^2/100} \quad \text{where } C = e^{C_1}$$

$$3 = Ce^0 \Rightarrow C = 3$$

$$Y = 3e^{t/5 - t^2/100}$$

(d) $\displaystyle\lim_{t\to\infty} 3e^{t/5 - t^2/100} = \lim_{t\to\infty} \frac{3e^{t/5}}{e^{t^2/100}} = \lim_{t\to\infty} \frac{3e^{t/5}}{(e^{t/5})^{t/20}} = 0$

Chapter 6 Key Terms

antidifferentiation by parts (p. 341)
antidifferentiation by substitution (p. 334)
arbitrary constant of integration (p. 331)
carbon-14 dating (p. 354)
carrying capacity (p. 367)
compounded continuously (p. 352)
constant of integration (p. 331)
continuous interest rate (p. 352)
decay constant (p. 351)
differential equation (p.321)
direction field (p. 323)
Euler's Method (p. 325)
evaluate an integral (p. 331)
exact differential equation (p. 321)
general solution to a differential equation (p. 321)
growth constant (p. 351)
first-order differential equation (p. 321)
first-order linear differential equation (p. 324)
graphical solution of a differential equation (p. 322)
half-life (p. 352)
Heaviside method (p. 363)
indefinite integral (p. 331)
initial condition (p. 321)
initial value problem (p. 321)
integral sign (p. 331)
integrand (p. 331)
integration by parts (p. 341)
Law of Exponential Change (p. 351)
Leibniz notation for integrals (p. 333)
logistic differential equation (p. 365)
logistic growth model (p. 367)
logistic regression (p. 365)
Newton's Law of Cooling (p. 354)
numerical method (p. 327)
numerical solution of a differential equation (p. 327)
order of a differential equation (p. 321)
partial fraction decomposition (p. 363)
particular solution (p. 321)
proper rational function (p. 363)
properties of indefinite integrals (p. 332)
radioactive (p. 352)
radioactive decay (p. 352)
resistance proportional to velocity (p. 360)
second-order differential equation (p. 330)
separable differential equations (p. 350)
separation of variables (p. 350)
slope field (p. 323)
solution of a differential equation (p. 321)
substitution in definite integrals (p. 336)
tabular integration (p. 344)
variable of integration (p. 331)

21. $\left(\frac{3\sin x}{10} - \frac{\cos x}{10}\right)e^{3x} + C$ **22.** $\left(-\frac{x^2}{3} - \frac{2x}{9} - \frac{2}{27}\right)e^{-3x} + C$ **24.** $\frac{1}{2}\ln\left|(2x-1)^3(x+1)^2\right| + C$

Chapter 6 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–10, evaluate the integral analytically. Then use NINT to support your result.

1. $\int_0^{\pi/3} \sec^2\theta\, d\theta$ $\sqrt{3}$

2. $\int_1^2 \left(x + \frac{1}{x^2}\right) dx$ 2

3. $\int_0^1 \frac{36\, dx}{(2x+1)^3}$ 8

4. $\int_{-1}^1 2x\sin(1-x^2)\, dx$ 0

5. $\int_0^{\pi/2} 5\sin^{3/2}x\cos x\, dx$ 2

6. $\int_{1/2}^4 \frac{x^2+3x}{x}\, dx$ $147/8$

7. $\int_0^{\pi/4} e^{\tan x}\sec^2 x\, dx$ $e-1$

8. $\int_1^e \frac{\sqrt{\ln r}}{r}\, dr$ $\frac{2}{3}$

9. $\int_0^1 \frac{x}{x^2+5x+6}\, dx$ $\ln(64/9) - \ln(27/4) = \ln(256/243)$

10. $\int_1^2 \frac{2x+6}{x^2-3x}\, dx$ $\ln(1/4) - \ln 16 = -6\ln 2$

In Exercises 11–24, evaluate the integral.

11. $\int \frac{\cos x}{2-\sin x}\, dx$ $-\ln|2-\sin x| + C$

12. $\int \frac{dx}{\sqrt[3]{3x+4}}$ $\frac{1}{2}(3x+4)^{2/3} + C$

13. $\int \frac{t\, dt}{t^2+5}$ $\frac{1}{2}\ln(t^2+5) + C$

14. $\int \frac{1}{\theta^2}\sec\frac{1}{\theta}\tan\frac{1}{\theta}\, d\theta$ $-\sec\frac{1}{\theta} + C$

15. $\int \frac{\tan(\ln y)}{y}\, dy$ $-\ln|\cos(\ln y)| + C)$

16. $\int e^x\sec(e^x)\, dx$ $\ln|\sec(e^x) + \tan(e^x)| + C$

17. $\int \frac{dx}{x\ln x}$ $\ln|\ln x| + C$

18. $\int \frac{dt}{t\sqrt{t}}$ $-\frac{2}{\sqrt{t}} + C$

19. $\int x^3\cos x\, dx$

20. $\int x^4\ln x\, dx$ $\frac{x^5\ln x}{5} - \frac{x^5}{25} + C$

21. $\int e^{3x}\sin x\, dx$

22. $\int x^2 e^{-3x}\, dx$

23. $\int \frac{25}{x^2-25}\, dx$ $\frac{5}{2}\ln\left|\frac{x-5}{x+5}\right| + C$

24. $\int \frac{5x+2}{2x^2+x-1}\, dx$

In Exercises 25–34, solve the initial value problem analytically. Support your solution by overlaying its graph on a slope field of the differential equation.

25. $\frac{dy}{dx} = 1 + x + \frac{x^2}{2}$, $y(0) = 1$ $y = \frac{x^3}{6} + \frac{x^2}{2} + x + 1$

26. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2$, $y(1) = 1$ $y = \frac{x^3}{3} + 2x - \frac{1}{x} - \frac{1}{3}$

27. $\frac{dy}{dt} = \frac{1}{t+4}$, $y(-3) = 2$ $y = \ln(t+4) + 2$

28. $\frac{dy}{d\theta} = \csc 2\theta\cot 2\theta$, $y(\pi/4) = 1$ $y = -\frac{1}{2}\csc 2\theta + \frac{3}{2}$

29. $\frac{d^2y}{dx^2} = 2x - \frac{1}{x^2}$, $x > 0$, $y'(1) = 1$, $y(1) = 0$ $y = \frac{x^3}{3} + \ln x - x + \frac{2}{3}$

30. $\frac{d^3r}{dt^3} = -\cos t$, $r''(0) = r'(0) = r(0) = -1$ $r = \sin t - \frac{t^2}{2} - 2t - 1$

31. $\frac{dy}{dx} = y + 2$, $y(0) = 2$ $y = 4e^x - 2$

32. $\frac{dy}{dx} = (2x+1)(y+1)$, $y(-1) = 1$ $y = 2e^{x^2+x} - 1$

33. $\frac{dy}{dt} = y(1-y)$, $y(0) = 0.1$ $y = \frac{1}{1+9e^{-t}}$

34. $\frac{dy}{dx} = 0.001y(100-y)$, $y(0) = 5$ $y = \frac{100}{1+19e^{-0.1x}}$

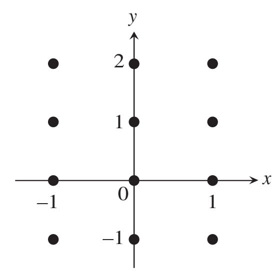
35. Find an integral equation $y = \int_a^x f(t)dt$ such that $dy/dx = \sin^3 x$ and $y = 5$ when $x = 4$. $y = \int_4^x \sin^3 t\, dt + 5$

36. Find an integral equation $y = \int_a^x f(t)dt$ such that $dy/dx = \sqrt{1+x^4}$ and $y = 2$ when $x = 1$. $y = \int_1^x \sqrt{1+t^4}\, dt + 2$

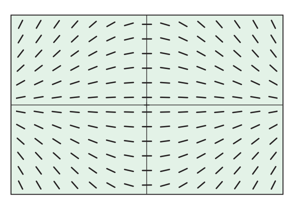
In Exercises 37 and 38, construct a slope field for the differential equation. In each case, copy the graph shown and draw tiny segments through the twelve lattice points shown in the graph. Use slope analysis, not your graphing calculator.

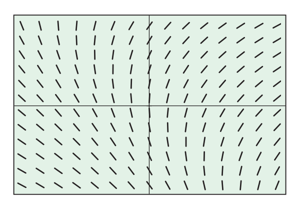
37. $\frac{dy}{dx} = -x$

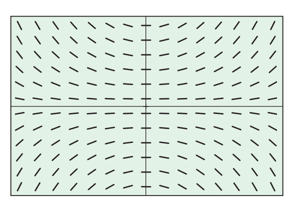
38. $\frac{dy}{dx} = 1 - y$

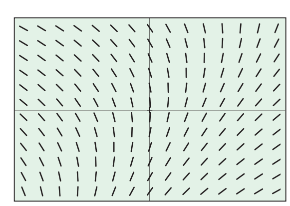


In Exercises 39–42, match the differential equation with the appropriate slope field. (All slope fields are shown in the window $[-6, 6]$ by $[-4, 4]$.)


(a)


(b)


(c)


(d)

39. $\frac{dy}{dx} = \frac{5}{x+y}$ Graph (b)

40. $\frac{dy}{dx} = \frac{5}{x-y}$ Graph (d)

41. $\frac{dy}{dx} = \frac{xy}{10}$ Graph (c)

42. $\frac{dy}{dx} = -\frac{xy}{10}$ Graph (a)

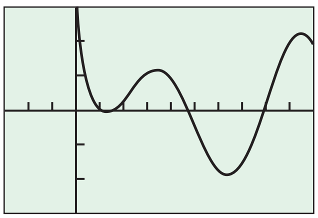
19. $x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C$

43. Suppose $dy/dx = x + y - 1$ and $y = 1$ when $x = 1$. Use Euler's Method with increments of $\Delta x = 0.1$ to approximate the value of y when $x = 1.3$. 1.362

44. Suppose $dy/dx = x - y$ and $y = 2$ when $x = 1$. Use Euler's Method with increments of $\Delta x = -0.1$ to approximate the value of y when $x = 0.7$. 2.362

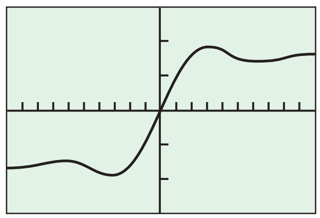
In Exercises 45 and 46, match the indefinite integral with the graph of one of the antiderivatives of the integrand.

45. $\int \frac{\sin x}{x}\, dx$ Graph (b)   **46.** $\int e^{-x^2}\, dx$ Graph (d)



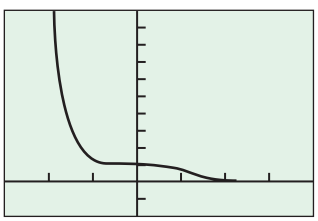
[–3, 10] by [–3, 3]

(a)



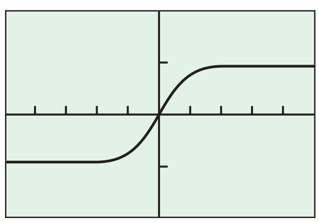
[–10, 10] by [–3, 3]

(b)



[–3, 4] by [–2, 10]

(c)



[–5, 5] by [–2, 2]

(d)

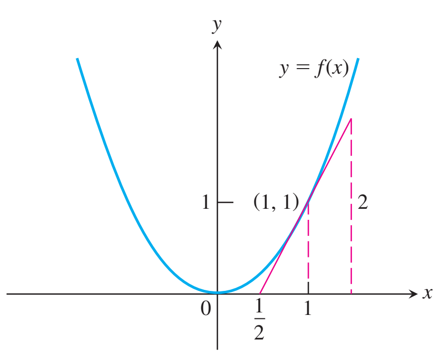
47. Writing to Learn The figure shows the graph of the function $y = f(x)$ that is the solution of one of the following initial value problems. Which one? How do you know? iv, since the given graph looks like $y = x^2$, which satisfies $dy/dx = 2x$ and $y(1) = 1$.

i. $dy/dx = 2x, \quad y(1) = 0$

ii. $dy/dx = x^2, \quad y(1) = 1$

iii. $dy/dx = 2x + 2, \quad y(1) = 1$

iv. $dy/dx = 2x, \quad y(1) = 1$



48. Writing to Learn Does the following initial value problem have a solution? Explain. Yes, $y = x$ is a solution.

$$\frac{d^2y}{dx^2} = 0, \quad y'(0) = 1, \quad y(0) = 0$$

49. Moving Particle The acceleration of a particle moving along a coordinate line is

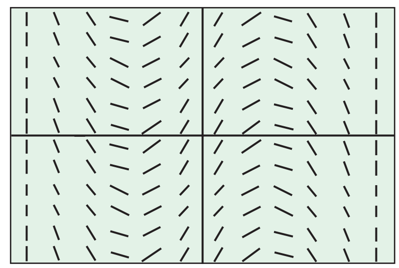
$$\frac{d^2s}{dt^2} = 2 + 6t \text{ m/sec}^2.$$

At $t = 0$ the velocity is 4 m/sec.

(a) Find the velocity as a function of time t. $v = 2t + 3t^2 + 4$

(b) How far does the particle move during the first second of its trip, from $t = 0$ to $t = 1$? 6 m

50. Sketching Solutions Draw a possible graph for the function $y = f(x)$ with slope field given in the figure that satisfies the initial condition $y(0) = 0$.



[–10, 10] by [–10, 10]

51. Californium-252 What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only about 8 g of it have been made in the western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per second. $k \approx 0.262059$

(a) What is the value of k in the decay equation for this isotope?

(b) What is the isotope's mean life? (See Exercise 19, Section 6.4.) About 3.81593 years

52. Cooling a Pie A deep-dish apple pie, whose internal temperature was 220°F when removed from the oven, was set out on a 40°F breezy porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F? About 92 minutes

53. Finding Temperature A pan of warm water (46°C) was put into a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 minutes after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was. −3°C

54. Art Forgery A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery? About 41.2 years

55. Carbon-14 What is the age of a sample of charcoal in which 90% of the carbon-14 that was originally present has decayed? About 18,935 years old

56. Appreciation A violin made in 1785 by John Betts, one of England's finest violin makers, cost \$250 in 1924 and sold for \$7500 in 1988. Assuming a constant relative rate of appreciation, what was that rate? About 5.3%

60. Use the Fundamental Theorem of Calculus to obtain $y' = \sin(x^2) + 3x^2 + 1$. Then differentiate again and also verify the initial conditions.

57. *Working Underwater* The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL,$$

where k is a constant. As a diver you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below a tenth of the surface value. About how deep can you expect to work without artificial light? About 59.8 ft

58. *Transport through a Cell Membrane* Under certain conditions, the result of the movement of a dissolved substance across a cell's membrane is described by the equation

$$\frac{dy}{dt} = k\frac{A}{V}(c - y),$$

where y is the concentration of the substance inside the cell, and dy/dt is the rate with which y changes over time. The letters k, A, V, and c stand for constants, k being the *permeability coefficient* (a property of the membrane), A the surface area of the membrane, V the cell's volume, and c the concentration of the substance outside the cell. The equation says that the rate at which the concentration changes within the cell is proportional to the difference between it and the outside concentration.

(a) Solve the equation for $y(t)$, using $y_0 = y(0)$. $y = c + (y_0 - c)e^{-(kA/V)t}$

(b) Find the steady-state concentration, $\lim_{t\to\infty} y(t)$. c

59. *Logistic Equation* The spread of flu in a certain school is given by the formula

$$P(t) = \frac{150}{1 + e^{4.3-t}},$$

where t is the number of days after students are first exposed to infected students.

(a) Show that the function is a solution of a logistic differential equation. Identify k and the carrying capacity. $k = 1$; carrying capacity $= 150$

(b) Writing to Learn Estimate $P(0)$. Explain its meaning in the context of the problem. ≈ 2; Initially there were 2 infected students.

(c) Estimate the number of days it will take for a total of 125 students to become infected. About 6 days

60. *Confirming a Solution* Show that

$$y = \int_0^x \sin(t^2)\,dt + x^3 + x + 2$$

is the solution of the initial value problem.

Differential equation: $y'' = 2x\cos(x^2) + 6x$

Initial conditions: $y'(0) = 1, \quad y(0) = 2$

61. *Finding an Exact Solution* Use analytic methods to find the exact solution to $P = \dfrac{800}{1 + 15e^{-0.002t}}$

$$\frac{dP}{dt} = 0.002P\left(1 - \frac{P}{800}\right), \quad P(0) = 50.$$

62. *Supporting a Solution* Give two ways to provide graphical support for the integral formula

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

63. *Doubling Time* Find the amount of time required for \$10,000 to double if the 6.3% annual interest is compounded **(a)** annually, **(b)** continuously. **(a)** About 11.3 years **(b)** About 11 years

64. *Constant of Integration* Let

$$f(x) = \int_0^x u(t)\,dt \quad \text{and} \quad g(x) = \int_3^x u(t)\,dt.$$

(a) Show that f and g are antiderivatives of $u(x)$.

(b) Find a constant C so that $f(x) = g(x) + C$.

65. *Population Growth* Table 6.9 shows the population of Anchorage, AK for selected years between 1950 and 2003.

Table 6.9 Population of Anchorage, AK

Years after 1950	Population
0	11,254
20	48,081
30	174,431
53	270,951

Source: Bureau of the Census, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2005.

(a) Use logistic regression to find a curve to model the data and superimpose it on a scatter plot of population against years after 1950.

(b) Based on the regression equation, what number will the Anchorage population approach in the long run?

(c) Write a logistic differential equation in the form $dp/dt = kP(M - P)$ that models the growth of the Anchorage data in Table 6.9.

(d) Writing to Learn The population of Anchorage in 2000 was 260,283. If this point is included in the data, how does it affect carrying capacity predicted by the regression equation? Is there reason to be concerned about our model?

66. *Temperature Experiment* A temperature probe is removed from a cup of hot chocolate and placed in water whose temperature (T_s) is 0°C. The data in Table 6.10 were collected over the next 30 sec with a CBL™ temperature probe.

Table 6.10 Experimental Data

Time t (sec)	T (°C)
2	74.68
5	61.99
10	34.89
15	21.95
20	15.36
25	11.89
30	10.02

(a) Find an exponential regression equation for the (t, T) data. Superimpose its graph on a scatter plot of the data.

(b) Estimate when the temperature probe will read 40°C.

(c) Estimate the hot chocolate's temperature when the temperature probe was removed.

AP* Examination Preparation

You may use a graphing calculator to solve the following problems.

67. The spread of a rumor through a small town is modeled by $dy/dt = 1.2y(1 - y)$, where y is the proportion of the townspeople who have heard the rumor at time t in days. At time $t = 0$, ten percent of the townspeople have heard the rumor.

(a) What proportion of the townspeople have heard the rumor when it is spreading the fastest? 1/2

(b) Find y explicitly as a function of t.

(c) At what time t is the rumor spreading the fastest?

68. A population P of wolves at time t years ($t \geq 0$) is increasing at a rate directly proportional to $600 - P$, where the constant of proportionality is k.

(a) If $P(0) = 200$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 500$, find k.

(c) Find $\lim_{t \to \infty} P(t)$.

69. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $dv/dt = -2(v + 17)$, with initial condition $v(0) = -47$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t \to \infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

67. (b) Separate the variables to get $\frac{dy}{y(1 - y)} = 1.2dt$. Solve the differential equation using the same steps as in Example 5 in Section 6.5 to obtain $y = \frac{1}{1 + 9e^{-1.2t}}$.

(c) Set $\frac{1}{2} = \frac{1}{1 + 9e^{-1.2t}}$ and solve for t to obtain $t = \frac{5 \ln 3}{3} \approx 1.83$ days.

Calculus at Work

I have a Bachelor's and Master's degree in Aerospace Engineering from the University of California at Davis. I started my professional career as a Facility Engineer managing productivity and maintenance projects in the Unitary Project Wind Tunnel facility at NASA Ames Research Center. I used calculus and differential equations in fluid mechanic analyses of the tunnels. I then moved to the position of Test Manager, still using some fluid mechanics and other mechanical engineering analysis tools to solve problems. For example, the lift and drag forces acting on an airplane wing can be determined by integrating the known pressure distribution on the wing.

I am currently a NASA On-Site Systems Engineer for the Lunar Prospector spacecraft project, at Lockheed Martin Missiles and Space in Sunnyvale, California. Differential equations and integration are used to design some of the flight hardware for the spacecraft. I work on ensuring that the different systems of the spacecraft are adequately integrated together to meet the specified design requirements. This often means doing some analysis to determine if the systems will function properly and within the constraints of the space environment. Some of these analyses require use of differential equations and integration to determine the most exact results, within some margin of error.

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